Characteristic Times in the motion of a particle

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G. Iannaccone, B. Pellegrini, *Characteristic Times in the motion of a particle* , Physical Review B, **49**, 23, pp.15468-15484 (1994).

PHYSICAL REVIEW B VOLUME 49, NUMBER 23 15 JUNE 1994-I

Characteristic times in the motion of a particle

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(Received 2 July 1993; revised manuscript received 19 January 1994)

Quantum mechanics does not provide direct tools for calculating time quantities related to the motion of a particle. In this paper we introduce a meaningful "time," the "stay time" in a space region, and we propose a method for calculating its statistical distribution. The stay time is obtained by a method based on Feynman's path integrals, which is similar to the one devised by Sokolovski and Baskin. We add a perturbative potential to the region being considered in order to induce variations in the wave function from which we can draw information about the time spent in the region. Unlike Sokolovski and Baskin, however, we obtain a real stay time and real greater order moments of its distribution. We also analyze other two "event times," the "time of presence" at a given position and the "time of passage" through a surface. These times, which were introduced by Olkhovski and Recami, are obtained directly from the time evolution of the probability density and the probability current density. We find some relations between such times and the stay time, which show the consistency of the proposed method. Our approach is internally self-consistent, allows a general analysis of the characteristic times in the motion of a quantum particle, and is effective in explaining the results of other studies, in particular in the field of the tunneling times of potential barriers.

I. INTRODUCTION

A variety of time quantities can be related to the motion of a classical particle; for instance, the time spent in a given region of space, and the moment at which the particle reaches a given position. Classical mechanics provides proper tools for obtaining these quantities.

On the contrary, in quantum mechanics time is a "dynamical parameter,"[1,2] so there is no direct tool (i.e., an operator) for determining the meaningful terms of the statistical distribution of time quantities. A proper solution to this problem is needed in the study of the high-frequency behavior of quantum devices, tunneling phenomena, and nuclear and chemical reactions.[3]

Interest in this subject has produced a considerable number of studies in recent years, particularly concerning the problem of tunneling times across potential barriers. A wide-ranging review of proposals concerning tunneling times can be found in the paper by Hauge and Støvneng,[4] and more recent studies can be found in the literature.[5–10]

However, no unified and general approach exists that allows us to obtain the desired times; rather, there is a variety of proposals leading to different results, even of orders of magnitude in the case of the tunnel effect. The only well-defined and well-established result is the "dwell time," which is the average time spent in a given region by all incoming particles.

The aim of this paper is to propose a method for defining and determining the average and greater order moments of the distribution of the time spent in a given region, which we call the "stay time." Moreover we will examine two "event times." Event times are the average instants of time at which an event occurs. We use a procedure of averaging upon times described by Olkhovski and Recami[11,12] for obtaining the mean time the particle is at a given position in space (which we call "time of presence"), and the mean time the particle traverses a given surface (which we call "time of passage").

Afterwards we obtain some relations between time of passage, stay time, and dwell time, which prove the consistency of our results and confirm the validity of the proposed method. Finally, we show that our approach is a suitable tool for explaining the results of other studies in the case of the tunnel effect.[13,14]

The method for obtaining the stay time basically consists in adding a perturbative potential to the region being considered and then analyzing how the wave function changes after interaction with it. In order not to modify the evolution of the wave function we have to make the potential approach zero and draw information from the derivative of the wave function with respect to the perturbation.

Certain approaches to the tunneling time problem consist basically in adding an infinitesimal perturbation to the barrier region: a magnetic field,[15–17] an oscillating potential,[14] or a constant potential.[18] Our method is closer to that of Sokolovski and Baskin,[18] in the sense that we use a constant and uniform perturbative potential and Feynman's path integral technique.[19] These authors obtain a complex time, whose physical meaning has been widely investigated;[5–7] Sokolovski and Connor[7] show that the complex nature of this time is a consequence of the uncertainty principle. We will analyze this point later.

On the basis of Sokolovski and Baskin's method, Fertig[8] has recently defined an amplitude distribution of "traversal times" in the case of a one-dimensional rectangular barrier. Then he obtains a complex mean traversal time and complex moments of the distribution, whose role seems to be uncertain.

Our method, which holds true for any system, leads to a more plausible real time; moreover, it allows us to calculate greater order moments of the distribution, notably the second-order moment, and through it, the standard deviation of the stay time. In particular we find that the mean stay time and its standard deviation coincide with the real and imaginary parts, respectively, of the complex time obtained by Sokolovski and Baskin.[18] Furthermore, we obtain a relation between the perturbative potential in the given region and the stay time in it, which is formally analogous to that connecting conjugate quantities, and, therefore, suggests the possibility of defining a stay time operator.

Finally, handling probability distribution functions straightforwardly suggests an extension of the probability interpretation that leads to the definition of the time of presence and of the time of passage. It has been shown that this extension leads to the definition of a time operator.[11]

Therefore, we feel that the statement concerning the impossibility of defining a time operator[20] deserves fresh consideration. New interest is now being shown in this topic.[21]

II. STAY TIME

A. Perturbative potential method

The time evolution of the wave function $\Psi_0(\mathbf{r},t)$ of an m-mass particle is determined by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi_0(\mathbf{r},t) = \hat{H}_0(\mathbf{r},t)\Psi_0(\mathbf{r},t), \tag{2.1}$$

where $\hat{H}_0(\mathbf{r},t)$ is the Hamiltonian operator

$$\hat{H}_0(\mathbf{r},t) = -\frac{\hbar^2\nabla^2}{2m} + V_0(\mathbf{r},t), \tag{2.2}$$

$V_0(\mathbf{r},t)$ being the potential experienced by the particle and $h = 2\pi\hbar$ being Planck's constant.

Our first aim is to determine the mean time spent in a given region Ω and in the time interval (t_0,t_1) by the particle found in $d\mathbf{r_1}$ at time $t = t_1$. In order to do this, at time $t = t_0$ we superimpose a uniform and time-independent perturbative potential V on the region Ω. The new Hamiltonian of the system becomes

$$\hat{H}(V,t_0;\mathbf{r},t) = \hat{H}_0(\mathbf{r},t) + Vu(t-t_0)\Theta_\Omega(\mathbf{r}), \tag{2.3}$$

where $\Theta_\Omega(\mathbf{r})$ equals 1 if $\mathbf{r} \in \Omega$ and 0 otherwise, and $u(t-t_0)$ equals 1 if $t \geq t_0$ and 0 otherwise.

From a physical point of view we might consider V as a testing potential applied in Ω from time t_0 for detecting the presence and then the stay time of the particle in Ω. Probably we could also use other quantities, such as a uniform magnetic field affecting the particle with spin and producing effects suitable for detecting the presence and the related stay time in the region being considered.

When we make the probe quantity approach zero, the outcoming time will be that of the unperturbed system. With the Hamiltonian of (2.3), the Schrödinger equation of the system becomes

$$i\hbar\frac{\partial}{\partial t}\Psi(V,t_0;\mathbf{r},t) = \hat{H}(V,t_0;\mathbf{r},t)\Psi(V,t_0;\mathbf{r},t), \tag{2.4}$$

where we have made the dependence of Ψ on V and t_0 explicit. For the following calculations and results it is more convenient to write the function in the exponential form

$$\Psi(V,t_0;\mathbf{r},t) = R(V,t_0;\mathbf{r},t)\exp\left[\frac{i}{\hbar}S(V,t_0;\mathbf{r},t)\right], \tag{2.5}$$

R and S being real functions.

Let us consider the particle described by Ψ_0, and found in $\mathbf{r_1}$ at time t_1. The results of the next section will enable us to state that the mean time it spends in Ω from t_0 to t_1 (from now on this will be referred to as the "stay time") is

$$\overline{\tau}(t_0;\mathbf{r_1},t_1) \equiv -\left.\frac{\partial S}{\partial V}\right|_{V=0} = \mathrm{Re}\left.\left\{\frac{i\hbar}{\Psi}\frac{\partial\Psi}{\partial V}\right\}\right|_{V=0}, \tag{2.6}$$

and the mean square stay time in Ω is

$$\begin{aligned}\overline{\tau^2}(t_0;\mathbf{r_1},t_1) &\equiv \left.\left(\frac{\partial S}{\partial V}\right)^2\right|_{V=0} + \left.\left(\frac{\hbar}{R}\frac{\partial R}{\partial V}\right)^2\right|_{V=0} \\ &= \left.\left|\frac{i\hbar}{\Psi}\frac{\partial\Psi}{\partial V}\right|^2\right|_{V=0},\end{aligned} \tag{2.7}$$

so that the standard deviation $\sigma_\tau(t_0;\mathbf{r_1},t_1)$ of the stay time in Ω becomes

$$\begin{aligned}\sigma_\tau(t_0;\mathbf{r_1},t_1) &= \{\overline{\tau^2}(t_0;\mathbf{r_1},t_1) \\ &\quad -[\overline{\tau}(t_0;\mathbf{r_1},t_1)]^2\}^{1/2} \\ &= \left|\left.\frac{\hbar}{R}\frac{\partial R}{\partial V}\right|_{V=0}\right|.\end{aligned} \tag{2.8}$$

B. Path integrals and stay time in a given region

The aim of this section is to show that an interesting relation exists between the perturbative potential on region Ω and the stay time in it. We refer to the situation described in (2.4).

Using Feynman's path integral technique[19] we can write, for all $\mathbf{r_1}$ and t_1,

$$\Psi(V,t_0;\mathbf{r_1},t_1) = \int k(V;\mathbf{r_1},t_1;\mathbf{r_0},t_0)\Psi(V,t_0;\mathbf{r_0},t_0)d\mathbf{r_0}, \tag{2.9}$$

in which the integral is over the whole space and the kernel $k(V;\mathbf{r_1},t_1;\mathbf{r_0},t_0)$ is defined as

$$k(V;\mathbf{r_1},t_1;\mathbf{r_0},t_0) \equiv \int \exp\left(\frac{i}{\hbar}\mathcal{S}[V,\mathbf{r}(\cdot)]\right)\mathcal{D}\mathbf{r}(\cdot), \quad (2.10)$$

where

$$\mathcal{S}[V,\mathbf{r}(\cdot)] \equiv \int_{t_0}^{t_1} [\tfrac{1}{2}m\dot{\mathbf{r}}^2(t) - V_0(\mathbf{r}(t),t) - Vu(t-t_0)\Theta_\Omega(\mathbf{r}(t))]dt. \quad (2.11)$$

Equation (2.10) is a *path integral*, i.e., a sum of terms of the type $\exp\{i\mathcal{S}[V,\mathbf{r}(\cdot)]/\hbar\}$ over all arbitrary space-temporal paths $\mathbf{r}(\cdot)$ in such a way that $\mathbf{r}(t_0) = \mathbf{r_0}$ and $\mathbf{r}(t_1) = \mathbf{r_1}$.

From time t_0 to t_1, $\mathbf{r}(t)$ is inside Ω for a time $\tau[\mathbf{r}(\cdot)]$ given by

$$\tau[\mathbf{r}(\cdot)] = \int_{t_0}^{t_1} \Theta_\Omega(\mathbf{r}(t))dt; \quad (2.12)$$

thus we can say that $\tau[\mathbf{r}(\cdot)]$ is the stay time in Ω corresponding to the path $\mathbf{r}(t)$; this value is clearly in the range $(0, t_1 - t_0)$. Equation (2.11) can now be written as

$$\mathcal{S}[V,\mathbf{r}(\cdot)] = \mathcal{S}[0,\mathbf{r}(\cdot)] - V\tau[\mathbf{r}(\cdot)]. \quad (2.13)$$

Let us substitute the expression just written in (2.10) and write it in double integral form, i.e.,

$$k(V;\mathbf{r_1},t_1;\mathbf{r_0},t_0) \equiv \int_0^{t_1-t_0} d\tau \int \exp\left(\frac{i}{\hbar}\mathcal{S}[0,\mathbf{r}_\tau(\cdot)] - i\frac{V\tau}{\hbar}\right)\mathcal{D}\mathbf{r}_\tau(\cdot), \quad (2.14)$$

where the second integral covers all the paths $\mathbf{r}_\tau$ whose stay time in Ω equals τ.

We can also write (2.14) in the form

$$k(V;\mathbf{r_1},t_1;\mathbf{r_0},t_0) = \int_0^{t_1-t_0} e^{-iV\tau/\hbar}\gamma(\tau;\mathbf{r_1},t_1;\mathbf{r_0},t_0)d\tau \quad (2.15)$$

if we define

$$\gamma(\tau;\mathbf{r_1},t_1;\mathbf{r_0},t_0) \equiv \int \exp\left(\frac{i}{\hbar}\mathcal{S}[0,\mathbf{r}_\tau(\cdot)]\right)\mathcal{D}\mathbf{r}_\tau(\cdot). \quad (2.16)$$

As we can see from (2.16), γ too is a path integral, corresponding to a null potential V, and obtained by summing over the paths whose stay time is τ.

By definition (2.12), τ is in the range $(0, t_1 - t_0)$, so if we choose $\tau \notin (0, t_1 - t_0)$ there is no corresponding path and the integral in (2.16) is over a null domain. Consequently the limits of integration in (2.15) can be extended from $-\infty$ to $+\infty$. Now (2.9) becomes

$$\Psi(V,t_0;\mathbf{r_1},t_1) = \int_{-\infty}^{+\infty} d\tau e^{-iV\tau/\hbar} \times \int \gamma(\tau;\mathbf{r_1},t_1;\mathbf{r_0},t_0) \times \Psi(V,t_0;\mathbf{r_0},t_0)d\mathbf{r_0}. \quad (2.17)$$

Before t_0, the instant we turn on the potential V, the Schrödinger equation describing the system is (2.1). Since the wave function is continuous with respect to time, we have $\Psi(V,t_0;\mathbf{r},t_0) = \Psi_0(\mathbf{r},t_0)$ and (2.17) becomes

$$\Psi(V,t_0;\mathbf{r_1},t_1) = \int_{-\infty}^{+\infty} e^{-iV\tau/\hbar}\Phi(\tau,t_0;\mathbf{r_1},t_1)d\tau, \quad (2.18)$$

where we have put

$$\Phi(\tau,t_0;\mathbf{r_1},t_1) \equiv \int \gamma(\tau;\mathbf{r_1},t_1;\mathbf{r_0},t_0)\Psi_0(\mathbf{r_0},t_0)d\mathbf{r_0}. \quad (2.19)$$

Equation (2.18) is a Fourier-transform relation in which the potential V superimposed on Ω and the stay time τ in Ω are conjugate quantities.

Within the framework of Feynman's approach, we can analyze (2.18) in a similar way to all the cases in which the wave function is given by a Fourier-transform-like expression, e.g., for a change of representation.[19] From this point of view $\Psi(V,t_0;\mathbf{r_1},t_1)$ is the probability amplitude that, when V is the perturbative potential on Ω, the particle is at position $\mathbf{r_1}$ at time t_1; it is a sum over different alternatives.

Each alternative can be written as $\Phi(\tau,t_0;\mathbf{r_1},t_1)\exp(-iV\tau/\hbar)$. The former term is the amplitude that the particle is in $\mathbf{r_1}$ at t_1, and has spent a time τ in Ω; the latter, $\exp(-iV\tau/\hbar)$, is the amplitude that, if the time spent in Ω is τ, then the perturbative potential is V.

C. Mean stay time in Ω

According to the above considerations, and to what we do when we deal with Fourier transforms, we could obtain a mean $\langle\tau(t_0;\mathbf{r_1},t_1)\rangle$ of τ for the wave function $\Psi_0(V,t_0;\mathbf{r_1},t_1)$ by simply using $|\Phi(\tau,t_0;\mathbf{r_1},t_1)|^2$ as a probability density of τ. So we could write

$$\langle\tau(t_0;\mathbf{r_1},t_1)\rangle \equiv \frac{\int_0^{t_1-t_0}\tau|\Phi(\tau,t_0;\mathbf{r_1},t_1)|^2d\tau}{\int_0^{t_1-t_0}|\Phi(\tau,t_0;\mathbf{r_1},t_1)|^2d\tau}. \quad (2.20)$$

We see that in (2.20) we have no information about the value of V. The reason is that V and τ are conjugate quantities with respect to a Fourier transform, so a mean of τ implies averaging on V from $-\infty$ to ∞.

In reality, we are interested in a mean of τ for a particular value of V, i.e., for $V = 0$, so we cannot use (2.20) in this form. First of all, we need to write (2.20) in a form in which we can show the operation of averaging over V; then we can try to draw up an expression to which we give the meaning of the average of τ for a given V, i.e., a "local" average of τ.

In order to follow such a method, from (2.18), by performing the inverse Fourier transform, we obtain

$$\Phi(\tau,t_0;\mathbf{r_1},t_1) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\Psi(V,t_0;\mathbf{r_1},t_1)e^{iV\tau/\hbar}dV, \quad (2.21)$$

and substitute it in (2.20), so that we can write

$$\langle\tau(t_0;\mathbf{r_1},t_1)\rangle = \frac{\int_{-\infty}^{+\infty}|\Psi(V,t_0;\mathbf{r_1},t_1)|^2\overline{\tau_V}(V,t_0;\mathbf{r_1},t_1)dV}{\int_{-\infty}^{+\infty}|\Psi(V,t_0;\mathbf{r_1},t_1)|^2dV}, \quad (2.22)$$

where

$$\overline{\tau_V}(V,t_0;\mathbf{r_1},t_1) \equiv -\frac{\partial S(V,t_0;\mathbf{r_1},t_1)}{\partial V}. \quad (2.23)$$

Now $\langle\tau(t_0;\mathbf{r_1},t_1)\rangle$ is expressed as an average of $\overline{\tau_V}(V,t_0;\mathbf{r_1},t_1)$ on varying V. The weight of each $\overline{\tau_V}$ is the probability density that the particle is in $\mathbf{r_1}$ at $t=t_1$, when the perturbative potential in Ω from t_0 is V. Therefore we can give $\overline{\tau_V}$ the meaning of average stay time in Ω of the particle found in $\mathbf{r_1}$ at time t_1, when the superimposed potential on Ω from t_0 is V.

What we are really concerned about is the mean stay time when V is null; in this case the significant quantity is $\overline{\tau_V}(0,t_0;\mathbf{r_1},t_1)$. Putting $\overline{\tau}(t_0;\mathbf{r_1},t_1) \equiv \overline{\tau_V}(0,t_0;\mathbf{r_1},t_1)$, from (2.23) we get (2.6).

It is worth noting that in (2.22) we can use any function $[\overline{\tau_V}+s(V)]$ in the place of $\overline{\tau_V}$, where $s(V)$ is any arbitrary function of V such as $\int_{-\infty}^{+\infty}|\Psi|^2 s(V)dV=0$. Therefore, the operation of giving $\overline{\tau_V}$ the meaning of average stay time in Ω for a given V, according to (2.23), may appear to be arbitrary, in the sense that there is not only one choice for the expression of the local mean $\overline{\tau_V}$ of τ. Nevertheless, the validity of our approach can be demonstrated by checking the consistency and the plausibility of our results; any further discussion on $s(V)$ is beyond the scope of the present work. The same considerations apply to the following derivation of the mean square stay time.

D. Mean square stay time in Ω

We obtain the mean square time by using a similar procedure. By starting from the expression

$$\langle\tau^2(t_0;\mathbf{r_1},t_1)\rangle = \frac{\int_0^{t_1-t_0}\tau^2|\Phi(\tau,t_0;\mathbf{r_1},t_1)|^2d\tau}{\int_0^{t_1-t_0}|\Phi(\tau,t_0;\mathbf{r_1},t_1)|^2d\tau}, \quad (2.24)$$

from (2.21) we obtain

$$\langle\tau^2(t_0;\mathbf{r_1},t_1)\rangle = \frac{\int_{-\infty}^{+\infty}|\Psi(V,t_0;\mathbf{r_1},t_1)|^2\overline{\tau_V^2}(V,t_0;\mathbf{r_1},t_1)dV}{\int_{-\infty}^{+\infty}|\Psi(V,t_0;\mathbf{r_1},t_1)|^2dV}, \quad (2.25)$$

where we have put

$$\overline{\tau_V^2}(V,t_0;\mathbf{r_1},t_1) \equiv \left(\frac{\partial}{\partial V}S(V,t_0;\mathbf{r_1},t_1)\right)^2 + \left(\frac{\hbar}{R(V,t_0;\mathbf{r_1},t_1)}\frac{\partial}{\partial V} \times R(V,t_0;\mathbf{r_1},t_1)\right)^2; \quad (2.26)$$

$\overline{\tau_V^2}(V,t_0;\mathbf{r_1},t_1)$ is the mean square time spent in Ω by the particle in $\mathbf{r_1}$ for $t=t_1$.

Defining $\overline{\tau^2}(t_0;\mathbf{r_1},t_1) \equiv \overline{\tau_V^2}(0,t_0;\mathbf{r_1},t_1)$, we can write (2.7) directly. Again, we wish to point out that the mean stay time and the mean square stay time depend upon the point of observation $\mathbf{r_1}$ and upon the time interval (t_0,t_1) during which the measurement is taken.

Schulman and Ziolkowski[22] find that the major contribution to the propagator through a potential barrier comes from a path corresponding to a purely imaginary time spent in the barrier. They connect the complex average tunneling time to the complex pole of a scattering matrix, whose real and imaginary parts define the location and the width of the resonance. As a consequence they interpret the imaginary time as the effective spread in significant tunneling times.

We wish to point out that their imaginary time is equal to the imaginary part of Sokolovski and Baskin's complex time,[18] and to the standard deviation of our stay time (2.8). Therefore Schulman and Ziolkowski's considerations are in agreement with our result. For practical opaque barriers, the real part of the complex traversal time is much smaller than the imaginary part, so the assumption that the major contribution comes from the imaginary part of the time is a good approximation.

E. Stay time and the uncertainty principle

In this section we want to show that the uncertainty principle does not force us to obtain a complex stay time, as many authors assert.[5–8] Complex quantities typically arise as a consequence of Feynman averaging.[19] From our point of view, rather, the path integral technique is a tool which is only used for obtaining and explaining (2.18), as we have shown, but is no longer employed for determining the average times. It would have to be so, in reality, because the complex times obtained by other approaches are not really manageable.

Now we are going to briefly describe the averages performed in the quoted papers in order to show the differences of our approach. We start by writing (2.18) for $V=0$, i.e.,

$$\Psi(0,t_0;\mathbf{r_1},t_1) = \int_{-\infty}^{+\infty}\Phi(\tau,t_0;\mathbf{r_1},t_1)d\tau; \quad (2.27)$$

the probability amplitude that the particle is in $\mathbf{r_1}$ at time t_1 is the sum of the probability amplitudes of interfering alternatives, each corresponding to a time τ spent in the region Ω by the particle found in $\mathbf{r_1}$ at t_1.

Actually, the complex time τ_c is straightforwardly obtained[5–8,18] by simply performing a weighted average over the alternatives, i.e., by means of the relation

$$\tau_c(t_0,\mathbf{r_1},t_1) = \frac{\int_0^{t_1-t_0}\tau\Phi(\tau,t_0;\mathbf{r_1},t_1)d\tau}{\int_0^{t_1-t_0}\Phi(\tau,t_0;\mathbf{r_1},t_1)d\tau}. \quad (2.28)$$

According to Feynman's formulation of the uncertainty principle,[19] we cannot determine the alternative taken by our process without destroying the interference between the alternatives. So the average (2.28) has to be complex valued, and cannot be used to predict the stay time of the particle.[7]

In order to obtain $\tau_c(t_0;\mathbf{r_1},t_1)$ in a simpler way, we have just to notice that the operation of adding a perturbative potential upon the region acts as a prism, allowing us to "see" the contribution of the interfering alternatives, and to obtain the probability amplitude of each τ by the means of the Fourier transform (2.21), by virtue of which Eq. (2.28) gives the complex value

$$\tau_c(t_0;\mathbf{r_1},t_1) = \left\{\frac{i\hbar}{\Psi}\frac{\partial\Psi}{\partial V}\right\}\bigg|_{V=0}. \tag{2.29}$$

Our idea is to consider (2.18) not just as a way for obtaining $\Phi(\tau,t_0;\mathbf{r_1},t_1)$ easier than the integration over an infinite number of paths, but to exploit the relation between V and τ, with reference to the relations between other conjugate quantities such as, for instance, position and momentum. In order to show this, let us consider the wave function $\Psi_0(\mathbf{r},t)$ and its Fourier transform in momentum representation $\varphi_0(\mathbf{p},t)$; we can write

$$\Psi_0(\mathbf{r},t) = \int_{-\infty}^{+\infty}\varphi_0(\mathbf{p},t)\exp\left(i\mathbf{p}\cdot\mathbf{r}/\hbar\right)d\mathbf{p}, \tag{2.30}$$

from which, for $\mathbf{r}=\mathbf{0}$, we have

$$\Psi_0(0,t) = \int_{-\infty}^{+\infty}\varphi_0(\mathbf{p},t)\,d\mathbf{p}, \tag{2.31}$$

according to which the probability amplitude that the particle is in $\mathbf{r}=\mathbf{0}$ at t is a sum of interfering alternatives, each corresponding to a momentum $\mathbf{p}$ of the particle in $\mathbf{r}=\mathbf{0}$ at t.

Now let us suppose that we want to know the average momentum of the particle in $\mathbf{r}=0$ at t. According to (2.31), the situation is the same encountered in (2.27). Feynman's averaging of $\mathbf{p}$, as a matter of fact, leads us to obtain a complex valued momentum $\overline{\mathbf{p}}_c$, i.e.,

$$\overline{\mathbf{p}}_c(0,t) = \frac{\int_{-\infty}^{+\infty}\mathbf{p}\,\varphi_0(\mathbf{p},t)\,d\mathbf{p}}{\int_{-\infty}^{+\infty}\varphi_0(\mathbf{p},t)\,d\mathbf{p}} = \left\{-\frac{i\hbar\nabla\Psi_0}{\Psi_0}\right\}\bigg|_{\mathbf{r}=\mathbf{0}}, \tag{2.32}$$

which is a result completely analogous to (2.29).

An interesting way to overcome this difficulty is to use the current probability density

$$\mathbf{J}(\mathbf{r},t) = \frac{1}{m}\mathrm{Re}\{\Psi_0^*(-i\hbar\nabla)\Psi_0\}. \tag{2.33}$$

This definition of $\mathbf{J}$ is largely accepted in quantum mechanics, even if it contains much that is arbitrary;[1] it is just the simplest real expression that obeys the continuity equation for the probability density. Moreover, (2.33) leads to a hydrodynamical definition of average momentum

$$\overline{\mathbf{p}}(0,t) \equiv m\frac{J(0,t)}{|\Psi_0(0,t)|^2} = \mathrm{Re}\left\{-\frac{i\hbar\nabla\Psi_0}{\Psi_0}\right\}\bigg|_{\mathbf{r}=\mathbf{0}}, \tag{2.34}$$

which has the undoubted merit of being real valued. Incidentally, it is interesting to notice that, in Bohm's interpretation of quantum mechanics,[23] $\overline{\mathbf{p}}$ defined in (2.34) is the actual momentum of the particle *being* in $\mathbf{r}=\mathbf{0}$ at t.

In our approach we exploit the relation between V and τ for obtaining (2.6), which is formally analogous to (2.34). Also in our case, owing to the undefined function $s(V)$ (see Sec. II C), (2.6) is somewhat arbitrary, and its validity can be checked by verifying the self-consistency of our results.

III. EVENT TIMES

In this section we are going to examine two other characteristic times, which are "event times," i.e., times at which something occurs. One is the average time at which a particle can be found at a given point, the "time of presence," the other is the average time at which a particle traverses a given surface, "the time of passage."

These times were introduced by Olkhovski and Recami[11,12] in conjunction with the introduction of a time operator in quantum mechanics. Regarding time as an observable leads to the possibility of defining expectation values and probability distributions for the time at which some event occurs.

A. Time of presence

The probability $P_A(t)$ that the particle described by $\Psi_0(\mathbf{r},t)$ is in a region A at time t is given by

$$P_A(t) \equiv \int_A |\Psi_0(\mathbf{r},t)|^2 d\mathbf{r}. \tag{3.1}$$

The question "when is the particle in A during the time interval (t_0,t_1)?" can only be answered by a statistical average

$$\overline{t}_A^P(t_0,t_1) \equiv \frac{\int_{t_0}^{t_1} tP_A(t)dt}{\int_{t_0}^{t_1} P_A(t)dt}. \tag{3.2}$$

If we substitute (3.1) in (3.2) and make A smaller and smaller, up to an infinitesimal volume $d\mathbf{r_1}$ around the position $\mathbf{r_1}$, we obtain

$$\overline{t}^P(t_0,t_1;\mathbf{r_1}) \equiv \frac{\int_{t_0}^{t_1} t|\Psi_0(\mathbf{r_1},t)|^2dt}{\int_{t_0}^{t_1}|\Psi_0(\mathbf{r_1},t)|^2dt}. \tag{3.3}$$

This expression can be regarded as the mean time at which the particle described by Ψ_0 can be found in $d\mathbf{r_1}$ in the time interval (t_0,t_1).

We call $\overline{t}^P(t_0,t_1;\mathbf{r_1})$ the mean "time of presence" in $\mathbf{r_1}$ in the interval (t_0,t_1). Equation (3.3) implies an extension of the probability interpretation, in the sense that $|\Psi_0(\mathbf{r},t)|^2d\mathbf{r}dt$ is the probability that the particle is in the volume $d\mathbf{r}$ in the time interval $(t,t+dt)$.

For a one-dimensional case (3.3) reduces to

$$\bar{t}^{P}(t_0,t_1;x_1) \equiv \frac{\int_{t_0}^{t_1} t|\Psi_0(x_1,t)|^2 dt}{\int_{t_0}^{t_1} |\Psi_0(x_1,t)|^2 dt}. \quad (3.4)$$

Definition (3.3) of the time of presence suggests the possibility of introducing a time operator; a wide study of this subject can be found in the papers by Olkhovski and Recami.[11,12]

B. Time of passage

The definition of time of passage requires another extension of the probability interpretation.[12] Given an infinitesimal surface dS, centered in $\mathbf{r_1}$ and perpendicular to the versor $\mathbf{n}$, the probability flux dF through dS at time t_1 is

$$dF(\mathbf{r_1},t_1) = \mathbf{J}(\mathbf{r_1},t_1)\cdot \mathbf{n} dS, \quad (3.5)$$

where the current probability density $\mathbf{J}$ is given by (2.33).

If all the components of Ψ_0 in the momentum representation give a contribution to the flux through dS at t_1 which is positive valued, i.e., the traversal of dS is possible only in the direction of $\mathbf{n}$, we can assume that the probability that the particle passes through dS in the time interval (t_1, t_1+dt) is

$$dP = dF(\mathbf{r_1},t_1)dt = \mathbf{J}(\mathbf{r_1},t_1)\cdot \mathbf{n}\, dS\, dt. \quad (3.6)$$

The probability that the particle traverses a surface Γ in the time interval t_1, t_1+dt is

$$P_\Gamma(t_1)dt = \int_\Gamma \mathbf{J}(\mathbf{r_1},t_1)\cdot \mathbf{n}\, dS\, dt, \quad (3.7)$$

and the mean time at which the particle traverses Γ during the time interval (t_0,t_1) is

$$\bar{t}^{\Gamma}(t_0,t_1) = \frac{\int_{t_0}^{t_1} tP_\Gamma(t)dt}{\int_{t_0}^{t_1} P_\Gamma(t)dt}. \quad (3.8)$$

Also in this case dP defined in (3.6) has to remain positive during (t_0,t_1) on every point of Γ. We call $\overline{t^{\Gamma}}$ the "time of passage" through the surface Γ. It seems to us a better name than "arrival time," used by other authors.[24]

For the one-dimensional case, the surface Γ reduces to a plane $x = x_1$ and the time of passage becomes

$$\bar{t}^{\Gamma}(t_0,t_1;x_1) = \frac{\int_{t_0}^{t_1} tJ(x_1,t)dt}{\int_{t_0}^{t_1} J(x_1,t)dt}. \quad (3.9)$$

The time of presence (3.3) and the time of passage (3.8), of course, are different quantities, from an intuitive point of view too. In the one-dimensional problem the difference is more subtle: a wave function component slower than the average has a greater weight in the calculation of the time of presence at a position x_1 because it is not vanishing for a long time in x_1; on the contrary, it has less weight in the time of passage through x_1, because only the probability current is involved. It can easily be shown that, if the spread in momentum of the wave function tends to zero, the difference between the times of presence and of passage correspondingly vanishes.

IV. CONSISTENCY CHECKS

A. Space partition

We are going to show an interesting relation between the stay times in different regions of space which is a first consistency check of our approach. Let us make a partition of the physical space in n regions Ω_j ($j = 1,\dots,n$) with no restrictions as to whether they are limited or unlimited (of course, at least one has to be unlimited). On each region Ω_j, at time t_0, we add a perturbative potential V_j; the particle now obeys the Schrödinger equation

$$\hat{H}_0\Psi + \sum_{j=1}^{n} V_j u(t-t_0)\Theta_{\Omega_j}(\mathbf{r})\Psi = i\hbar\frac{\partial}{\partial t}\Psi. \quad (4.1)$$

In this case Ψ depends on each V_j, but we wish to avoid explicitly stating this.

On the basis of previous results, i.e., of (2.6), if the particle is in $\mathbf{r_1}$ at time t_1, its mean stay time in Ω_j ($j = 1,\dots,n$) from time t_0 to t_1 is

$$\bar{\tau}_j(t_0;\mathbf{r_1},t_1) = \mathrm{Re}\left\{\frac{i\hbar}{\Psi}\frac{\partial\Psi}{\partial V_j}\right\}\Bigg|_{\substack{V_l=0\\ l=1,\dots,n}}. \quad (4.2)$$

Since we have (see Appendix A)

$$\sum_{j=1}^{n} i\hbar\frac{\partial\Psi}{\partial V_j} = (t_1-t_0)u(t_1-t_0)\Psi, \quad (4.3)$$

we can divide each term by Ψ and take only the real part, so that we obtain

$$\sum_{j=1}^{n}\mathrm{Re}\left\{\frac{i\hbar}{\Psi}\frac{\partial\Psi}{\partial V_j}\right\} = (t_1-t_0)u(t_1-t_0). \quad (4.4)$$

Evaluating this expression for $V_j = 0$, $j = 1,\dots,n$, and for $t_1 > t_0$, from (4.2) we obtain

$$t_1 = t_0 + \sum_{j=1}^{n}\bar{\tau}_j(t_0;\mathbf{r_1},t_1); \quad (4.5)$$

that is, the time elapsed from t_0 to t_1 is equal to the sum of the stay times spent in each region into which the whole space is partitioned by the particle found in $\mathbf{r_1}$ at t_1. Equation (4.5) is a first fundamental consistency check for our approach.

B. Relation between stay time and time of passage

From (4.5) we can derive a relation between the time of passage and the stay time that will be useful in the study of tunneling times. Let us consider an infinitesimal surface dS centered in $\mathbf{r_1}$ and orthogonal to versor $\mathbf{n}$. We have seen that $dP = \mathbf{J}(\mathbf{r_1},t)\cdot\mathbf{n}\,dS\,dt$ is the probability that the particle traverses dS in the time interval $(t, t+$

dt). If we want to know the mean stay time $\overline{\tau}^{\Gamma}_{j_\infty}(t_0)$ in Ω_j of a particle passing through a surface Γ from t_0 onwards we have to perform an integration on time and on Γ, so that we have

$$\overline{\tau}^{\Gamma}_{j_\infty}(t_0) = \frac{\int_{t_0}^{\infty} \int_{\Gamma} \overline{\tau}_j(t_0;\mathbf{r_1},t)\mathbf{J}(\mathbf{r_1},t)\cdot\mathbf{n}\,dS\,dt}{\int_{t_0}^{\infty} \int_{\Gamma} \mathbf{J}(\mathbf{r_1},t)\cdot\mathbf{n}\,dS\,dt}, \tag{4.6}$$

and, from (3.7), (3.8), (4.5), and (4.6) we can write

$$\overline{t}^{\Gamma}(t_0,\infty) = t_0 + \sum_{j=1}^{n} \overline{\tau}^{\Gamma}_{j_\infty}(t_0); \tag{4.7}$$

then the mean time of passage through Γ equals t_0 plus the sum of the stay times from t_0 onwards in each region into which the space is partitioned.

For a one-dimensional space, Γ reduces to the plane $x = x_1$, so that we have

$$\overline{t}^{\Gamma}(t_0,\infty;x_1) = t_0 + \sum_{j=1}^{n} \overline{\tau}_{j_\infty}(t_0,x_1), \tag{4.8}$$

where

$$\overline{\tau}_{j_\infty}(t_0,x_1) = \frac{\int_{t_0}^{\infty} \overline{\tau_j}(t_0;x_1,t)J(x_1,t)dt}{\int_{t_0}^{\infty} J(x_1,t)dt}. \tag{4.9}$$

C. Relation between stay time and dwell time

In the case of a stationary wave function, the time spent by a particle in a region Ω was introduced by Smith[25] as the ratio between the probability of finding the particle in Ω and the incoming probability flux. Büttiker,[17] in the one-dimensional case [when Ω is the interval (a,b)], defined this time, usually referred to as the "dwell time," as

$$\tau_{Db} \equiv \frac{1}{J}\int_a^b |\Psi_0(x)|^2 dx, \tag{4.10}$$

where J is the incident probability flux. Several authors[26,27] extended the definition of dwell time to the case in which Ψ_0 is a wave packet,

$$\tau_{D_{1\mathrm{dim}}}(t_0,t_1) \equiv \int_{t_0}^{t_1} dt \int_a^b |\Psi_0(x,t)|^2 dx; \tag{4.11}$$

where Ψ_0 now is normalized and $\tau_{D_{1\mathrm{dim}}}$ is the time spent in (a,b) from time t_0 to t_1 .

We can now straightforwardly generalize $\tau_{D_{1\mathrm{dim}}}$ to the three-dimensional case in the form

$$\tau_D(t_0,t_1) \equiv \int_{t_0}^{t_1} dt \int_{\Omega} |\Psi_0(\mathbf{r},t)|^2 d\mathbf{r}. \tag{4.12}$$

The aim of this section is to show that the stay time of (2.6) is consistent with the dwell time in (4.12).

If the particle is found in $\mathbf{r_1}$ at time t_1, it stayed in Ω for a time $\overline{\tau}(t_0;\mathbf{r_1},t_1)$ given by (2.6). If we do not know where the particle is at time t_1, the mean time spent in Ω is an average $\overline{\tau_{\mathrm{all}}}(t_0,t_1)$ of $\overline{\tau}(t_0;\mathbf{r_1},t_1)$ over all values of $\mathbf{r_1}$, weighted on the probability that the particle is in $\mathbf{r_1}$, i.e.,

$$\overline{\tau_{\mathrm{all}}}(t_0,t_1) = \frac{\int \overline{\tau}(t_0;\mathbf{r_1},t_1)|\Psi_0(\mathbf{r_1},t_1)|^2 d\mathbf{r_1}}{\int |\Psi_0(\mathbf{r_1},t_1)|^2 d\mathbf{r_1}}, \tag{4.13}$$

where the integrals are over the whole space, in order to cover all possible positions occupied by the particle.

Now let us show that (4.13) is equal to (4.12). The denominator of (4.13) equals 1 because Ψ_0 is normalized. Then from (2.6) we obtain

$$\overline{\tau_{\mathrm{all}}}(t_0,t_1) = -\int |\Psi(0,t_0;\mathbf{r_1},t_1)|^2 \times \left.\frac{\partial S(V,\mathbf{r_1},t_1)}{\partial V}\right|_{V=0} d\mathbf{r_1}. \tag{4.14}$$

The integral in (4.14) is over the whole space and Ψ is absolutely summable, so

$$\overline{\tau_{\mathrm{all}}}(t_0,t_1) = \int \Psi^*(0,t_0;\mathbf{r_1},t_1)i\hbar \times \left.\frac{\partial \Psi(V,t_0;\mathbf{r_1},t_1)}{\partial V}\right|_{V=0} d\mathbf{r_1}. \tag{4.15}$$

The perturbation theory applied to the path integral technique gives[19]

$$\Psi(V,t_0;\mathbf{r_1},t_1) = \Psi(0,t_0;\mathbf{r_1},t_1) - i\frac{V}{\hbar}\int_{t_0}^{t_1} dt \times \int_{\Omega} d\mathbf{r} k(0;\mathbf{r_1},t_1;\mathbf{r},t) \times \Psi(0,t_0;\mathbf{r},t) + O(V^2), \tag{4.16}$$

where $O(V^2)$ is a second-order infinitesimal with respect to V, so that we have

$$\left.\frac{\partial}{\partial V}\Psi(V,t_0;\mathbf{r_1},t_1)\right|_{V=0} = -\frac{i}{\hbar}\int_{t_0}^{t_1} dt \int_{\Omega} d\mathbf{r} k(0;\mathbf{r_1},t_1;\mathbf{r},t)\Psi(0,t_0;\mathbf{r},t). \tag{4.17}$$

Substituting this expression in (4.15) we can write

$$\overline{\tau_{\mathrm{all}}}(t_0,t_1) = \int d\mathbf{r_1}\Psi^*(0,t_0;\mathbf{r_1},t_1)\int_{t_0}^{t_1} dt \times \int_{\Omega} d\mathbf{r} k(0;\mathbf{r_1},t_1;\mathbf{r},t)\Psi(0,t_0;\mathbf{r},t). \tag{4.18}$$

Equation (4.18) leads straightforwardly to (4.12), i.e., $\overline{\tau_{\mathrm{all}}} = \tau_D$, if we remember that[19]

$$\int d\mathbf{r_1}\Psi^*(0,t_0;\mathbf{r_1},t_1)k(0;\mathbf{r_1},t_1;\mathbf{r},t) = \Psi^*(0,t_0;\mathbf{r},t), \tag{4.19}$$

and $\Psi(0,t_0,\mathbf{r},t) = \Psi_0(\mathbf{r},t)$. This is a further check of the validity of our method.

For subsequent applications, it should be noted that, if we are interested in a mean $\tau_A(t_0,t_1)$ of $\tau(t_0;\mathbf{r}_1,t_1)$ restricted to a region A of possible positions where we can find the particle at time t_1, in the place of (4.13) we have

$$\overline{\tau_A}(t_0,t_1) = \frac{\int_A \overline{\tau}(t_0;\mathbf{r}_1,t_1)|\Psi_0(\mathbf{r}_1,t_1)|^2 d\mathbf{r}_1}{\int_A |\Psi_0(\mathbf{r}_1,t_1)|^2 d\mathbf{r}_1}. \quad (4.20)$$

V. CHARACTERISTIC TUNNELING TIMES

A. Reflection and transmission times

On the basis of the results of previous sections, we are able to define and calculate the tunneling time of a potential barrier. To this end let us consider the one-dimensional problem, with a time-independent potential $V_0(x)$, nonvanishing only for $0 < x < d$, of the type sketched in Fig. 1.

Let the normalized wave function $\Psi_0(x,t)$ of the particle at time t_0 be substantially confined in $x < 0$, i.e., $\int_{-\infty}^0 |\Psi_0(x,t_0)|^2 dx \approx 1$, and let it be a wave packet whose components are moving to the right with an energy that is lower than the barrier height. After t_0, the packet moves closer to the barrier, undergoes scattering, and splits into a packet transmitted across the barrier and into a reflected one.

The probability $T_{t_1}(t_1)$ that at a time t_1 the particle is found to have crossed the barrier is

$$T_{t_1}(t_1) = \int_d^\infty |\Psi_0(x,t_1)|^2 dx, \quad (5.1)$$

the probability $R_{t_1}(t_1)$ that the particle is found to have been reflected back by the barrier is

$$R_{t_1}(t_1) = \int_{-\infty}^0 |\Psi_0(x,t_1)|^2 dx, \quad (5.2)$$

and, finally, the probability $I_{t_1}(t_1)$ of finding the particle inside the barrier is given by

$$I_{t_1}(t_1) = \int_0^d |\Psi_0(x,t_1)|^2 dx. \quad (5.3)$$

Since Ψ_0 is normalized, we have

$$R_{t_1}(t_1) + I_{t_1}(t_1) + T_{t_1}(t_1) = 1. \quad (5.4)$$

Let Ω be the barrier region $(0 < x < d)$; the time $\overline{\tau}(t_0;x,t_1)$, given by (2.6), is the time spent in the barrier region by the particle found in x at time t_1. If position x is on the right of the barrier, the particle has crossed the barrier; if x is on the left, the particle has been reflected back.

The average $\tau_T(t_0,t_1)$ of $\overline{\tau}(t_0;x,t_1)$ over all possible positions on the right of the barrier, corresponding to a particle which has crossed the barrier at t_1, according to (4.20) is

$$\tau_T(t_0,t_1) \equiv \frac{\int_d^\infty \overline{\tau}(t_0;x,t_1)|\Psi_0(x,t_1)|^2 dx}{\int_d^\infty |\Psi_0(x,t_1)|^2 dx}, \quad (5.5)$$

while the average $\tau_R(t_0,t_1)$ over all positions on the left of the barrier, corresponding to a particle which has been reflected back at t_1, is

$$\tau_R(t_0,t_1) \equiv \frac{\int_{-\infty}^0 \overline{\tau}(t_0;x,t_1)|\Psi_0(x,t_1)|^2 dx}{\int_{-\infty}^0 |\Psi_0(x,t_1)|^2 dx}. \quad (5.6)$$

The mean stay time in $(0,d)$ of a particle still in $(0,d)$ is

$$\tau_I(t_0,t_1) \equiv \frac{\int_0^d \overline{\tau}(t_0;x,t_1)|\Psi_0(x,t_1)|^2 dx}{\int_0^d |\Psi_0(x,t_1)|^2 dx}. \quad (5.7)$$

From (4.13) and (5.3)–(5.7), we obtain

$$\overline{\tau_{\rm all}}(t_0,t_1) = R_{t_1}(t_1)\tau_R(t_0,t_1) + I_{t_1}(t_1)\tau_I(t_0,t_1) + T_{t_1}(t_1)\tau_T(t_0,t_1). \quad (5.8)$$

In the case of $t_1 \to \infty$, since the wave function is not spatially confined, we have that $I_{t_1}(t_1)$ and $I_{t_1}(t_1)\tau_I(t_0,t_1)$ go to zero (see Appendix B).

Then, by defining $R = \lim_{t_1\to\infty} R_{t_1}(t_1)$ and $T = \lim_{t_1\to\infty} T_{t_1}(t_1)$, (5.4) and (5.8) become, respectively,

$$R + T = 1, \quad (5.9)$$

$$\tau_D(t_0,\infty) = \overline{\tau_{\rm all}}(t_0,\infty) = R\tau_R(t_0,\infty) + T\tau_T(t_0,\infty), \quad (5.10)$$

where we have taken into account the previous result according to which $\tau_D(t_0,t_1) = \overline{\tau_{\rm all}}(t_0,t_1)$. We call $\tau_R(t_0,\infty)$ and $\tau_T(t_0,\infty)$ reflection and transmission times, respectively, while R and T are the reflection and transmission probabilities, respectively.

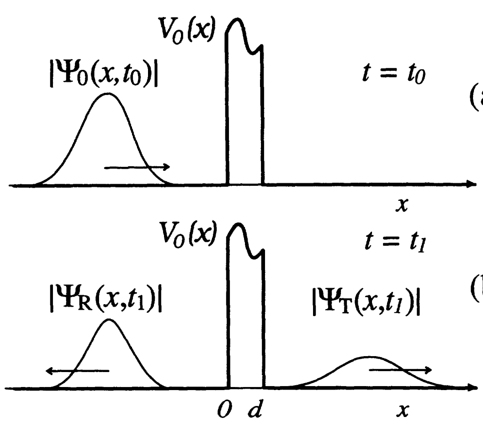

FIG. 1. (a) At time t_0 the particle is described by a wave packet which is substantially confined in $x < 0$ and moves to the right. (b) At time t_1 the interaction between the wave function and the barrier is substantially exhausted and we have a transmitted packet in $x > d$ and a reflected packet in $x < 0$.

Condition (5.10) is usually considered to be a crucial consistency check for any definition of tunneling time.[4,26,28] However, some criticism of this argument can come if one does not accept the dwell time as the mean time spent in the barrier by all incoming particles,[29] or if one does not consider tunneling and reflection times as meaningful concepts, and, at least in principle within conventional interpretations of quantum mechanics, as measurable quantities.[9,30] As we have shown in this section, as well as in Secs. II and IV C, this is not our case.

Moreover, (5.10) is not based on a decomposition of the probability density inside the barrier into "to be transmitted" and "to be reflected" components, which is considered impossible within conventional interpretations of quantum mechanics[24] or responsible for the presence of nonclassical interference terms.[31] On the contrary, the dwell time and the tunneling and reflection times are obtained by proper integration in space of the stay time [based upon (4.13) and (4.20)] at a time t_1 which tends to infinity; at that time we can certainly distinguish between "been trasmitted" and "been reflected" components of the wave function (in fact they are separated in space, i.e., the trasmitted component is in $x > d$, the reflected one in $x < 0$), so it is straightforward to obtain the decomposition (5.10) of the dwell time.

Equation (5.10) actually is a probabilistic condition; the particle crosses the barrier with probability T, and can be reflected with probability $R = 1-T$; in the former case it spends in Ω a time τ_T, in the latter case a time τ_R. Equation (5.10) gives the time spent in Ω from t_0 onwards averaged over all particles by simply applying the laws of conditional probability.

B. Are reflection and transmission times meaningful quantities?

Now we wish to address a question that involves the uncertainty principle and transmission and reflection times. Dumont and Marchioro[9] assert that the uncertainty principle does not allow us to determine separate transmission and reflection times, because their calculation involves the simultaneous measurement of noncommuting observables.

This claim, indeed, is based upon the fact that, when they try to decompose the dwell time corresponding to a given energy in terms of an outgoing-channel-specific basis, they obtain a cross term [see (14) of Ref. 9]. The cross term, as a matter of fact, is just an evidence that their particular attempt of decomposing the dwell time fails. However, this result is not general and, therefore, one cannot deduce from it that any other decomposition has to fail.

Moreover, Dumont and Marchioro assert that their candidates for transmission and reflection times "cannot have the desired interpretation because they are not specific to an initial state incoming from the left." So, other candidates could be well suited for obtaining the desired decomposition. Our definitions of transmission and reflection times, as a matter of fact, are specific to a particle incoming from the left and allow us to write the required decomposition of the dwell time (5.10) into separate transmission and reflection times.

C. Different approaches to tunneling time

In this section we examine two methods for calculating the tunneling times that have been proposed by other authors and that lead to different conclusions. Nevertheless, they come within the framework of our approach.

1. Phase time

The phase-time method was proposed by Wigner[32] and by Hartman[13] and has been widely reexamined.[4,33] The situation is the one shown in the previous section: the wave packet, moving to the right and still in $x < 0$ at time t_0, in the course of time, say t_1, breaks up into a transmitted packet [referred to as $\Psi_T(x_1,t_1)$] and a reflected packet [referred to as $\Psi_R(x_1,t_1)$]. If there were no barrier at all we should have a wave function, say Ψ_{inc}, which is the initial packet still freely moving toward the right.

The phase-time method consists basically in defining the tunneling time as $\tau_\Phi = t_{\text{out}} - t_{\text{in}}$, where t_{in} is the time at which Ψ_{inc} reaches position $x = 0$ [Fig. 2(a)] and t_{out} is the time at which Ψ_T reaches $x = d$ [Fig. 2(b)]. The instant $\tilde{t}(\tilde{x})$ at which a wave packet reaches a given position $\tilde{x}$ is determined in one of the following ways:

(i) as the time at which the peak of the packet is in $\tilde{x}$; this is the original idea of phase time;[13,32]

(ii) as the time at which the center of mass of the wave packet is in $\tilde{x}$;[33]

(iii) as the time of passage through position $\tilde{x}$ defined by (3.9).

The original idea (i) of phase time works for packets which are narrow enough in energy to make the

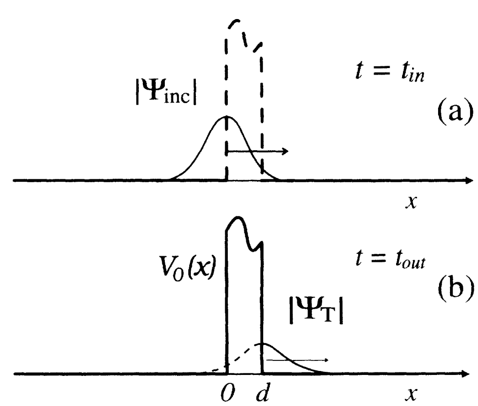

FIG. 2. The phase-time method consists basically in defining the tunneling time as $\tau_\phi = t_{\text{out}} - t_{\text{in}}$, where (a) t_{in} is the time at which the packet would be in $x = 0$ if there were no barrier at all, and (b) t_{out} is the time at which the transmitted packet would be in $x = d$.

stationary-phase approximation suitable. It has been demonstrated[33] that in this situation method (ii) produces the same results as (i). Now we shall show that, in the same case, the time of passage method (iii) gives equal results with (i) and (ii).

Let us write the wave function at time t_0 in the form

$$\Psi_0(x,t_0) = \int g(E-\tilde{E})e^{ip(x-x_0)/\hbar}dE, \tag{5.11}$$

where $p = (2mE)^{1/2}$, $\tilde{E}$ is the mean value of energy, and $g(E-\tilde{E})$ is a real function that is nonvanishing only if $|E-\tilde{E}|$ is small enough to keep the stationary-phase approximation valid. These initial conditions give

$$\Psi_{\rm inc}(x,t) = \int g(E-\tilde{E})e^{ip(x-x_0)/\hbar} \times e^{-iE(t-t_0)/\hbar}dE, \tag{5.12}$$

while the transmitted packet is

$$\Psi_T(x,t) = \int g(E-\tilde{E})a(E)e^{ip(x-x_0-d)/\hbar} \times e^{-iE(t-t_0)/\hbar}dE, \tag{5.13}$$

for $x > d$, $a(E)$ being the transmission coefficient for energy E.

Now $t_{\rm in}$ is the mean time of passage at $x = 0$ from t_0 onwards, i.e., from (3.9),

$$t_{\rm in} = \bar{t}^{\Gamma}_{\rm inc}(t_0,\infty;0) = \frac{\int_{t_0}^{\infty} tJ_{\rm inc}(0,t)dt}{\int_{t_0}^{\infty} J_{\rm inc}(0,t)dt}, \tag{5.14}$$

where $J_{\rm inc}$ is the probability current density of $\Psi_{\rm inc}$. Substituting (5.12) in this expression yields

$$t_{\rm in} = \frac{\int g^2(E-\tilde{E})\,[t_0p(E)/m - x_0]\,dE}{\int g^2(E-\tilde{E})[p(E)/m]\,dE} \approx t_0 - \frac{x_0}{v}, \tag{5.15}$$

where $v = p(\tilde{E})/m$ is the average velocity of the wave packet.

Similarly, for Ψ_T, we obtain

$$t_{\rm out} = \bar{t}^{\Gamma}_T(t_0,\infty;d) \approx \hbar\left.\frac{\partial\alpha}{\partial E}\right|_{E=\tilde{E}} - \frac{x_0}{v} + t_0, \tag{5.16}$$

where $\alpha(E)$ is the argument of $a(E)$. Therefore, from (5.15) and (5.16), the phase time becomes

$$\tau_\Phi = t_{\rm out} - t_{\rm in} \approx \hbar\left.\frac{\partial\alpha}{\partial E}\right|_{E=\tilde{E}}, \tag{5.17}$$

i.e., by means of method (iii), we can obtain the phase time found by other authors.[4,13,32,33]

It should be noted that $t_{\rm in}$ and $t_{\rm out}$ are calculated in different conditions: for $t_{\rm in}$ we have a free-moving packet, while for $t_{\rm out}$ we must consider the presence of the barrier.

If one identifies the phase time with the actual tunneling time, one implicitly assumes that the mean time at which a particle crossing the barrier reaches $x = 0$ is equal to the one that we obtain when no barrier exists; in other words the barrier in $0 < x < d$ would not affect particle motion for $x < 0$. Our approach makes it possible to show the error of this assumption. Let us split the x axis into three regions: Ω_1 $(x < 0)$, Ω_2 $(0 \le x \le d)$, and Ω_3 $(x > d)$. Let us consider the particle described by $\Psi_{\rm inc}$; let $\bar{\tau}^{\rm inc}_{1\infty}$ be the stay time in Ω_1 averaged as in (4.9), and $\bar{\tau}^{\rm inc}_{2\text{-}3\infty}$ the stay time in the region made by the union of Ω_2 and Ω_3 (i.e., $x > 0$).

From (4.8) we obtain

$$\begin{aligned} t_{\rm in} &= \bar{t}^{\Gamma}_{\rm inc}(t_0,\infty;0) \\ &= t_0 + \bar{\tau}^{\rm inc}_{1\infty}(t_0;0) + \bar{\tau}^{\rm inc}_{2\text{-}3\infty}(t_0;0). \end{aligned} \tag{5.18}$$

Remembering that $\Psi_{\rm inc}$ represents a wave packet freely moving toward the right with the center of mass in $x = x_0$ at time t_0, we easily obtain

$$\bar{\tau}^{\rm inc}_{1\infty}(t_0,0) \approx -\frac{x_0}{v} \tag{5.19}$$

and

$$\bar{\tau}^{\rm inc}_{2\text{-}3\infty}(t_0,0) = 0. \tag{5.20}$$

Now consider Ψ_T: from (4.8) we get

$$\begin{aligned} t_{\rm out} = \bar{t}^{\Gamma}_T(t_0,\infty;d) &= t_0 + \bar{\tau}^T_{1\infty}(t_0,d) \\ &+ \bar{\tau}^T_{2\infty}(t_0,d) + \bar{\tau}^T_{3\infty}(t_0,d), \end{aligned} \tag{5.21}$$

where $\bar{\tau}^T_{j\infty}$ denotes the stay time in Ω_j averaged as in (4.9).

The mean stay time in the barrier of the tunneling particles is $\bar{\tau}^T_{2_\infty}(t_0,d)$. It is the tunneling time, and it is easy to verify that in our case it equals the value given by (5.5) when t_1 tends to infinity.

From (4.9) and (5.13) we obtain

$$\bar{\tau}^T_{1_\infty}(t_0,d) \approx -\frac{x_0}{v} - \hbar\left.\frac{\partial\alpha}{\partial V_1}(\tilde{E})\right|_{V_1=V_2=V_3=0}, \tag{5.22}$$

$$\bar{\tau}^T_{2_\infty}(t_0,d) \approx -\hbar\left.\frac{\partial\alpha}{\partial V_2}(\tilde{E})\right|_{V_1=V_2=V_3=0}, \tag{5.23}$$

$$\bar{\tau}^T_{3_\infty}(t_0,d) \approx -\hbar\left.\frac{\partial\alpha}{\partial V_3}(\tilde{E})\right|_{V_1=V_2=V_3=0}. \tag{5.24}$$

From (5.19) and (5.22) we see that the times spent in Ω_1 in the two different situations are not the same. The difference between them is

$$\begin{aligned} \Delta\tau_1 &= \bar{\tau}^T_{1\infty}(t_0,d) - \bar{\tau}^{\rm inc}_{1\infty}(t_0,0) \\ &= -\hbar\left.\frac{\partial\alpha}{\partial V_1}(E)\right|_{V_1=V_2=V_3=0} \end{aligned} \tag{5.25}$$

Finally we have

$$\tau_\Phi = \bar{\tau}^T_{2\infty}(t_0,d) + \Delta\tau, \tag{5.26}$$

with

$$\Delta\tau = \Delta\tau_1 + \bar{\tau}^T_{3\infty}(t_0,d). \tag{5.27}$$

The phase time is not equal to the tunneling time; rather, it is the delay in the motion of the particle due to the presence of the barrier. A part of this delay is the actual tunneling time (that spent in the barrier region); the other part is the one represented by $\Delta\tau$.

This additional delay has also been found in other studies,[4,26] in expressions linking together the phase time and the dwell time [for instance (4.4) of Ref. 4], and has been referred to as *self-interference delay.* Our approach makes it possible to split this delay into a component $\Delta\tau_1$ corresponding to a time spent in the region before the barrier, and a component $\overline{\tau}^T_{3_\infty}(t_0, d)$ corresponding to a time spent in the region behind the barrier.

2. The oscillating barrier

This method was introduced by Büttiker and Landauer.[14,34,35] Let us consider the potential outlined in Fig. 1 and let the interval $(0, d)$ be the region Ω. The idea of Büttiker and Landauer consists in superimposing an oscillating potential $V_1 \cos\omega t$ on the region Ω.

The total potential is

$$V(x,t) = V_0(x) + V_1 \Theta_\Omega(x) \cos\omega t. \tag{5.28}$$

The incident particle is expressed as a monochromatic wave function of energy E.

As a consequence of barrier modulation the resulting wave function includes sidebands with energy $E \pm n\hbar\omega$. The components of energy $E + n\hbar\omega$ correspond to the absorption of n modulation quanta by the particle, while those of energy $E - n\hbar\omega$ correspond to the emission of n quanta. To first order in V_1 only the components E, $E + \hbar\omega$, $E - \hbar\omega$ are meaningful.

Let us use $T_\pm$ to indicate the intensity of the transmitted components of energy $E \pm \hbar\omega$, and T to indicate that of the component of energy E. The ratio between these components, to first order in V_1 and for ω approaching zero, is

$$\frac{T_\pm}{T} = \left(\frac{V_1 \tau_{\rm BL}}{2\hbar}\right)^2, \tag{5.29}$$

where

$$\tau_{\rm BL} = \hbar \left| \frac{1}{\Psi} \frac{\partial \Psi}{\partial V_1} \right|_{V_1=0}. \tag{5.30}$$

The equations are taken from (3.7) and (4.4) in Ref. 34.

Büttiker and Landauer state that $\tau_{\rm BL}$ is the characteristic time of the interaction between the particle and the barrier and identify it with the tunneling time. It should be noted that the square of $\tau_{\rm BL}$ exactly equals the mean square stay time in the barrier region given by (2.7), considering that the dependence of $\overline{\tau^2}$ on t_0, $\mathbf{r}_1$, and t_1 vanishes for monochromatic wave functions. Moreover, the square of $\tau_{\rm BL}$ appears in (5.29), so that it seems more plausible to give $\tau^2_{\rm BL}$ the meaning of mean square tunneling time rather than to identify $\tau_{\rm BL}$ with the mean tunneling time.

VI. OPEN PROBLEMS

In the case of tunneling through a potential barrier, our approach leads to the same tunneling time as Rybachenko's method.[15,16] As has been noticed,[28] this time depends slowly on the barrier thickness so that, if the barrier is thick enough, we have mean traversal velocities larger than the speed of light. No postulate has been violated as long as we deal with nonrelativistic quantum mechanics.

In order to obtain an expression for the stay time in a relativistic framework, we have to use relativistic path integrals, in such a way as to have a new expression in place of (2.18). Unfortunately, the extension of Feynman's method to the Hartree equation or to the Dirac equation needs the introduction of a fifth parameter (besides position and time) that complicates the formalism and makes us no longer able to associate a stay time in a given region to the four-dimensional paths obtained (position and time as a function of the fifth parameter).

The problem needs deeper analysis. It is undoubted that we cannot simply substitute the transmission amplitude obtained from the Dirac equation in expression (2.6) of the stay time, as Leavens and Aers[28] have done. Equation (2.6), indeed, has been derived from (2.18), whose validity must be verified on the basis of relativistic path integrals.

It could also be possible that velocities greater than the speed of light in the computation of barrier traversal are an effect of quantum nonlocality.[36] Further study is needed in this direction.

Another problem has been pointed out by Leavens and Aers[28] for the case of the Larmor clock method, but affects our approach also. In order to show it let us refer to the situation of Fig. 1: the point is that if we calculate the time spent by a reflected particle in a region situated to the right of the barrier, we obtain a time which is not zero. If on the right of the barrier the potential is zero everywhere, common sense suggests that a particle that spends some time in a region situated to the right of the barrier will continue its motion toward the right, and will not be reflected back.

It should be noticed that, when we apply the perturbative potential to the region, we introduce a discontinuity in the potential at the frontier of the region. Even when the perturbative potential is zero, the field is still infinite, because the potential varies by an infinitesimal quantity over a distance which is exactly zero.

There is the possibility that our problem is just a side effect of the discontinuities we introduced; in other words, that particles that have traversed the barrier are then reflected back by these potential discontinuities, and that this effect is not negligible if the actual potential of the region is uniform (i.e., there are no other causes of scattering). However, we agree with Leavens and Aers[28] when they say that further study of Wigner trajectories (and perhaps also Bohm trajectories) will decide if this problem is another example of quantum nonlocality, or if the Larmor clock approach (and our approach) is applicable to a narrower extent in the case of tunneling. Up to the present it seems that Wigner trajectories are not useful for this problem.[37] Bohm trajectories have been extensively studied,[38] and it appears that there is no reentrant trajectory; nevertheless, it could be interesting to see if an infinitesimal potential on the far side of the barrier might significantly perturb electron trajectories.

VII. CONCLUSIONS

The method which we have proposed allows a general analysis of characteristic time quantities in the motion of a quantum particle. These quantities have only a statistical meaning and our method makes it possible to define and calculate any order moment of their distribution. We have shown that the stay time approach gives results which are totally consistent with the dwell time and other event times, in particular the time of presence and the time of passage, and, moreover, it is a suitable tool for explaining and analyzing characteristic times obtained with other methods.

In detail we have applied our method to the tunneling time problem. The time we obtain satisfies the consistency requirements established by Hauge and Støvneng[4]; it is a real quantity and complies with the required probabilistic condition. We have shown that there is no need of considering complex valued averages, whose role is still unclear. Furthermore we have shown that the time of passage can be used for obtaining the phase time and to see that the phase time is not the tunneling time, rather it is the total delay in the wave packet motion due to the presence of the barrier, of which the time spent inside the barrier is merely a part.

The possibility of calculating the mean square tunneling time sheds light on the meaning of Büttiker and Landauer's time $\tau_{\rm BL}$ obtained with the oscillating barrier method. In our opinion it is more plausible to consider $\tau_{\rm BL}^2$ the mean square tunneling time rather than $\tau_{\rm BL}$ the mean tunneling time. This opinion is supported by the fact that Büttiker and Landauer give a definition of $\tau_{\rm BL}^2$ and not of $\tau_{\rm BL}$.

Our method for defining and calculating the stay time rests on Feynman's path integral approach, as does the method by Solokovski and Baskin.[18] These authors obtain a complex time whose real part equals the mean stay time and whose imaginary part equals the standard deviation of our stay time.

The Larmor clock method introduced by Rybachenko[15] and Baz[16] gives a tunneling time equal to the one found by us, but it does not explain the results obtained by using other methods. Moreover, the only attempt at using the Larmor clock method for calculating the second moment of the distribution, which was made by Baz', is relevant to a very particular case of unity reflection probability and of a time distribution which is a δ function.

The general and versatile nature of the method described here makes it reliable and useful for several applications, and it is effective as a framework for including and/or explaining time quantities reported in other studies.

ACKNOWLEDGMENTS

The present work has been supported by the Ministry for the University and Scientific and Technological Research, by the Italian National Research Council (CNR), and in particular by the CNR Finalized Project "Materials and Devices for Solid-State Electronics."

APPENDIX A

We can rewrite (4.1) in the form

$$\hat{H}_p \Psi = i\hbar \frac{\partial}{\partial t}\Psi, \tag{A1}$$

where

$$\hat{H}_p = \hat{H}_0 + \sum_{j=1}^{n} V_j u(t-t_0)\Theta_{\Omega_j}(\mathbf{r}). \tag{A2}$$

If at time t_0 we impose a uniform potential W over the whole space, (A1) becomes

$$\left[\hat{H}_p + Wu(t-t_0)\right]\varphi = i\hbar\frac{\partial}{\partial t}\varphi, \tag{A3}$$

of which we now seek a solution of the type

$$\varphi = \Psi f, \tag{A4}$$

where $f(W,t)$ is a function depending upon W and t alone; moreover we need $f(W,t_0) = 1 \quad (\forall W)$, because W is imposed at time t_0 and $\varphi(t_0) = \Psi(t_0)$.

Substituting (A4) in (A3) we obtain

$$f\hat{H}_p\Psi + Wu(t-t_0)f\Psi = i\hbar f\frac{\partial}{\partial t}\Psi + i\hbar\Psi\frac{\partial}{\partial t}f, \tag{A5}$$

and then from (A1) we get

$$Wu(t-t_0)f = i\hbar\frac{\partial}{\partial t}f. \tag{A6}$$

The solution of (A6), with the initial conditon $f(W,t_0) = 1$, is

$$f(W,t) = \exp\left[-\frac{i}{\hbar}W(t-t_0)u(t-t_0)\right]. \tag{A7}$$

Therefore, superimposing a potential W over the whole space has the same effect as adding a quantity W to the potential V_j in each region partitioning the space; that is, from (A4) and (A7) we obtain

$$\begin{aligned}\varphi &= \Psi(V_1+W,\ldots,V_n+W)\\ &= \exp\left[-\frac{i}{\hbar}W(t-t_0)u(t-t_0)\right]\Psi(V_1,\ldots,V_n),\end{aligned} \tag{A8}$$

from which we finally get (4.3) in the form

$$\frac{\partial\varphi}{\partial W} = \sum_{j=1}^{n}\frac{\partial\Psi}{\partial V_j} = -\frac{i}{\hbar}u(t-t_0)(t-t_0)\Psi. \tag{A9}$$

APPENDIX B

For the potential sketched in Fig. 1 the wave function is not spatially confined, so that it has a continuous spectrum of energy eigenvalues, and it can be written in the form

$$\Psi_0(x,t) = \int_{-\infty}^{+\infty} \Phi(x,E) e^{i\frac{Et}{\hbar}} dE, \tag{B1}$$

where $\Phi(x,E)$ is limited $\forall x$. In our case we have wave functions with a finite energy band, i.e., with $\Phi(x,E)=0$ for $E > [V_0(x)]_{\max}$, so that

$$\int_{-\infty}^{+\infty} |\Phi(x,E)|^2 dE = P, \tag{B2}$$

where P is a real positive number.

The Parseval theorem now implies that

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dt = 2\pi\hbar P, \qquad \forall x. \tag{B3}$$

The convergence of this integral implies that $|\Psi(x,t)|^2$ goes to zero as $t \to \pm\infty$ more rapidly than $1/t$, i.e.,

$$\lim_{t_1 \to \pm\infty} |\Psi(x,t_1)|^2 t_1 = 0. \tag{B4}$$

On the basis of the definition (5.3) of $I_{t_1}(t_1)$ and considering that $0 \le \tau_I(t_0,t_1) \le t_1 - t_0$, we finally obtain the relations

$$\lim_{t_1 \to \pm\infty} I_{t_1}(t_1) = 0, \tag{B5}$$

and

$$\lim_{t_1 \to \pm\infty} I_{t_1}(t_1)\tau_I(t_0,t_1) = 0, \tag{B6}$$

that have been used to write (5.9) and (5.10).

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