Theory of conductance and noise additivity in parallel mesoscopic conductors

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Theory of conductance and noise additivity in parallel mesoscopic conductors

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We present a theory of conductance and noise in generic mesoscopic conductors connected in parallel, and we demonstrate that the additivity of conductance and of shot noise arises as a sole property of the junctions connecting the two (or more) conductors in parallel. Consequences on the functionality of devices based on the Aharonov-Bohm effect are also drawn. [S0163-1829(97)07643-1]

Electron transport is said to be in the ballistic regime when the phase coherence of the wave functions is maintained throughout the whole extension of the structure, in other words, when the inelastic mean free path is greater than the size of the device.

Recent advances in nanofabrication techniques, and the possibility of reaching operating temperatures in the millikelvin range (in order to suppress phonon scattering), have made feasible the study of transport in the ballistic regime, and the exploration of the interesting phenomena that emerge when such regime is approached.

Since the pioneering work of van Wees and co-workers on conductance quantization,[1] this transport regime has been the subject of widespread interest, both from a theoretical and an experimental point of view.

In this paper, we focus on transport and noise in mesoscopic conductors connected in parallel. It suffices to point out that such topology applies to all devices based on some kind of Aharonov-Bohm effect.

In the case of macroscopic conductors connected in parallel it is well known that both the conductances and the shot-noise current power spectral densities add.

While transport properties of macroscopic conductors depend on local material properties, those of mesoscopic structures are obtained as the solution of a complex scattering problem, in which the shape of the boundaries and the potential profile over the whole device region play a relevant role. Therefore, the problem needs to be reformulated in these terms.

Numerical studies of conductance additivity in sample structures made of two parallel constrictions, along with some analytical justifications, exist in the literature.[2–4] Furthermore, a numerical study has been presented showing additivity of shot noise in parallel constrictions.[5] In addition, experimental results on the conductance of two or more parallel quantum point contacts are available.[6–9]

We present a theory of conductance and noise in generic mesoscopic conductors connected in parallel, and we demonstrate that the additivity of conductance and of shot noise arises as a sole property of the junctions connecting the two (or more) conductors in parallel.

In particular, additivity requires that the scattering matrix of both junctions [each of which is a $(n+1)$-lead mesoscopic system, if the whole device is made up of n conductors in parallel], is such that zero conductance appears between the leads connecting different conductors in parallel. In such a case, even if phase coherence is maintained throughout the whole device, there are no quantum-interference interactions between parallel conductors.

With the purpose of showing that this requirement is not hard to meet, we end the paper including the example of a junction warranting additivity. We also point out the consequences of our results on evaluating the functionality and the noise properties of devices based on the Aharonov-Bohm effect.

The structure considered is sketched in Fig. 1. It consists of two mesoscopic conductors Σ^u and Σ^d connected in parallel by means of two junctions, Σ^l and Σ^r. The junctions are ballistic systems with three leads, one connected to Σ^u, one to Σ^d, and the other used as an external lead of the whole structure. Phase coherence is mantained in the whole system.

The internal and external leads of the whole structure are numbered from 1 to 6, as sketched in Fig. 1. Let us define $\mathbf{a}_i$ and $\mathbf{b}_i$ $(i=1,\ldots,6)$ as the column vectors whose N_i elements are the amplitudes of the modes in lead i entering and exiting the adjacent junction, respectively.

Electron transport in each of the subsystems Σ^α $(\alpha=u,d,l,r)$ is completely described by the associated scattering matrix S^α,[10] which is unitary and such as

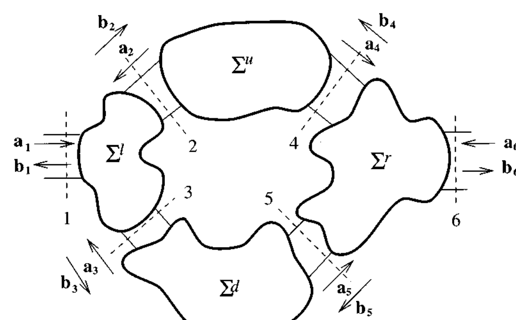

FIG. 1. The structure consists of two mesoscopic conductors Σ^u and Σ^d connected in parallel by means of two junctions, Σ^l and Σ^r. The junctions are ballistic systems with three leads, one connected to Σ^u, one to Σ^d, and the other used as an external lead of the whole structure. Phase coherence is mantained in the whole system. $\mathbf{a}_i$ and $\mathbf{b}_i$ $(i=1,\ldots,6)$ are the column vectors whose N_i elements are the amplitudes of the modes in lead i entering and exiting the adiacent junction, respectively.

$S^{\alpha T}(\mathbf{B})=S^{\alpha}(-\mathbf{B})$, where $\mathbf{B}$ is the applied magnetic field.[11]

Let us first consider the case in which we have only the left junction Σ^l and leads 1, 2, and 3 are connected to different electron reservoirs. The relation between incoming and outgoing modes can be written as

$$\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{bmatrix}=S^l\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix} \tag{1}$$

if S^l is arranged in the following way:

$$S_l=\begin{bmatrix} s_{11}^l & s_{12}^l & s_{13}^l\\ s_{21}^l & s_{22}^l & s_{23}^l\\ s_{31}^l & s_{32}^l & s_{33}^l\end{bmatrix}, \tag{2}$$

where s_{ij}^l $(i,j=1,2,3)$ is a $N_i\times N_j$ matrix, relating the amplitudes of the outgoing modes in lead i to the amplitudes of the incoming modes in lead j (as many evanescent modes as needed may be considered).

We can repeat the same considerations for the right junction: if leads 4, 5, and 6 are connected to different electron reservoirs we can write

$$\begin{bmatrix}\mathbf{b}_4\\ \mathbf{b}_5\\ \mathbf{b}_6\end{bmatrix}=\begin{bmatrix} s_{44}^r & s_{45}^r & s_{46}^r\\ s_{54}^r & s_{55}^r & s_{56}^r\\ s_{64}^r & s_{65}^r & s_{66}^r\end{bmatrix}\begin{bmatrix}\mathbf{a}_4\\ \mathbf{a}_5\\ \mathbf{a}_6\end{bmatrix}, \tag{3}$$

where we have already written S^r in the form of submatrices s_{ij}^r, $(i,j=4,5,6)$. Analogously, for the conductors Σ^u and Σ^d we can write

$$\begin{bmatrix}\mathbf{a}_2\\ \mathbf{a}_4\end{bmatrix}=S^u\begin{bmatrix}\mathbf{b}_2\\ \mathbf{b}_4\end{bmatrix}=\begin{bmatrix} s_{22}^u & s_{24}^u\\ s_{42}^u & s_{44}^u\end{bmatrix}\begin{bmatrix}\mathbf{b}_2\\ \mathbf{b}_4\end{bmatrix}, \tag{4}$$

$$\begin{bmatrix}\mathbf{a}_3\\ \mathbf{a}_5\end{bmatrix}=S^d\begin{bmatrix}\mathbf{b}_3\\ \mathbf{b}_5\end{bmatrix}=\begin{bmatrix} s_{33}^d & s_{35}^d\\ s_{53}^d & s_{55}^d\end{bmatrix}\begin{bmatrix}\mathbf{b}_3\\ \mathbf{b}_5\end{bmatrix}. \tag{5}$$

We intend to demonstrate that conductances and noise in parallel conductors add if the scattering matrices of the junctions satisfy the following conditions:

$$s_{32}^l=0,\quad s_{45}^r=0,$$

$$s_{23}^l=0,\quad s_{54}^r=0. \tag{6}$$

Let us point out that in the absence of magnetic field the last two conditions are redundant. The physical meaning of Eq. (6) is that an electron injected in Σ^l from lead 3 does not exit from lead 2 (and vice versa), and that an electron injected in Σ^r from lead 5 is not transmitted to lead 4 (and vice versa).

Since $S^{l\dagger}S^l=1$ and $S^{r\dagger}S^r=1$ (i.e., the S matrices are unitary), by multiplying row by column and using Eq. (6), we straightforwardly obtain

$$s_{21}^{l\dagger}s_{31}^l=0,\quad s_{21}^l s_{31}^{l\dagger}=0, \tag{7}$$

$$s_{64}^{r\dagger}s_{65}^r=0,\quad s_{64}^r s_{65}^{r\dagger}=0. \tag{8}$$

In order to proceed with our demonstration we have to consider the whole structure, and the associated scattering matrix S_{tot}. We can write

$$\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_6\end{bmatrix}=S_{\mathrm{tot}}\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_6\end{bmatrix}=\begin{bmatrix} s_{11} & s_{16}\\ s_{61} & s_{66}\end{bmatrix}\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_6\end{bmatrix}. \tag{9}$$

In addition, we have to consider the scattering matrices S_{up} and S_{down}, corresponding to the whole system with the conductor Σ^d or Σ^u removed, respectively. They are of the form

$$S_{\mathrm{up}}=\begin{bmatrix} s_{11}^u & s_{16}^u\\ s_{61}^u & s_{66}^u\end{bmatrix},\quad S_{\mathrm{down}}=\begin{bmatrix} s_{11}^d & s_{16}^d\\ s_{61}^d & s_{66}^d\end{bmatrix}. \tag{10}$$

Let us start by calculating s_{61}^u, which relates the amplitudes of the modes exiting from lead 6 to those of the modes entering from lead 1: since Σ^d has been removed the only path from lead 1 to lead 6 is that through the upper conductor; moreover, the hypothesis (6) implies that the terminations of leads 3 and 5 have no effect on the calculation of s_{61}^u. It is straighforward to write s_{61}^u in the form of the scattering series

$$\begin{aligned} s_{61}^u &= s_{64}^r s_{42}^u s_{21}^l + s_{64}^r s_{42}^u s_{22}^l s_{24}^u s_{44}^r s_{42}^u s_{21}^l+\cdots\\ &\quad + s_{64}^r s_{42}^u (s_{22}^l s_{24}^u s_{44}^r s_{42}^u)^n s_{21}^l\\ &= s_{64}^r s_{42}^u (1-s_{22}^l s_{24}^u s_{44}^r s_{42}^u)^{-1} s_{21}^l. \end{aligned} \tag{11}$$

Analogously, we can write s_{61}^d as

$$s_{61}^d = s_{65}^r s_{53}^d (1-s_{33}^l s_{35}^d s_{55}^r s_{53}^d)^{-1} s_{31}^l. \tag{12}$$

From Eqs. (7), (8), (11), and (12) we obtain

$$s_{61}^d s_{61}^{u\dagger}=0,\quad s_{61}^{d\dagger} s_{61}^u=0. \tag{13}$$

The calculation of s_{61} is simplified by the fact that the conditions (6) are such that an electron passing through Σ^d cannot be scattered into Σ^u, and vice versa. Therefore, we just have

$$s_{61}=s_{61}^u+s_{61}^d. \tag{14}$$

The conductance of the whole structure at 0 K is evaluated according to Landauer and Büttiker:[12–14]

$$G=\frac{2e^2}{h}\,\mathrm{tr}\{s_{61}^\dagger s_{61}\}. \tag{15}$$

Let G_{up} and G_{down} be the conductances of the system with the conductor Σ^d or Σ^u removed, respectively: we have

$$G_{\mathrm{up}}=\frac{2e^2}{h}\,\mathrm{tr}\{s_{61}^{u\dagger}s_{61}^u\},\quad G_{\mathrm{down}}=\frac{2e^2}{h}\,\mathrm{tr}\{s_{61}^{d\dagger}s_{61}^d\}. \tag{16}$$

From Eqs. (13) and (14) we have

$$s_{61}^\dagger s_{61}=(s_{61}^{u\dagger}+s_{61}^{d\dagger})(s_{61}^u+s_{61}^d)=s_{61}^{u\dagger}s_{61}^u+s_{61}^{d\dagger}s_{61}^d, \tag{17}$$

which allows us to write, from Eqs. (15) and (16),

$$G=G_{\mathrm{up}}+G_{\mathrm{down}}. \tag{18}$$

The additivity of conductances for parallel mesoscopic conductors has been demonstrated. It implies also the additivity of thermal noise current spectral densities, which are proportional to the conductance by means of a factor $4k_BT$, where k_B is Boltzmann's constant and T is the absolute temperature.

The shot-noise current spectral density in mesoscopic systems can be expressed in terms of the scattering matrix as derived by Büttiker:[15,16]

$$\langle(\Delta I)^2\rangle = 4|eV|\frac{e^2}{h}\,\mathrm{tr}\{s_{11}^{\dagger}s_{11}s_{61}^{\dagger}s_{61}\}$$

$$= 4|eV|\frac{e^2}{h}(\mathrm{tr}\{s_{61}^{\dagger}s_{61}\} - \mathrm{tr}\{s_{61}^{\dagger}s_{61}s_{61}^{\dagger}s_{61}\}) \quad (19)$$

where V is the voltage applied between leads 1 and 6, and the last equality comes from the unitarity of the matrix S_{tot}.

The shot-noise current spectral densities $\langle(\Delta I_{\mathrm{up}})^2\rangle$ and $\langle(\Delta I_{\mathrm{down}})^2\rangle$ of the system with Σ^d or Σ^u removed, respectively, are

$$\langle(\Delta I_{\mathrm{up}})^2\rangle = 4|eV|\frac{e^2}{h}(\mathrm{tr}\{s_{61}^{u\dagger}s_{61}^{u}\} - \mathrm{tr}\{s_{61}^{u\dagger}s_{61}^{u}s_{61}^{u\dagger}s_{61}^{u}\})$$

$$\langle(\Delta I_{\mathrm{down}})^2\rangle = 4|eV|\frac{e^2}{h}(\mathrm{tr}\{s_{61}^{d\dagger}s_{61}^{d}\} - \mathrm{tr}\{s_{61}^{d\dagger}s_{61}^{d}s_{61}^{d\dagger}s_{61}^{d}\}). \quad (20)$$

From Eqs. (13) and (14) we have

$$s_{61}^{\dagger}s_{61}s_{61}^{\dagger}s_{61} = s_{61}^{u\dagger}s_{61}^{u}s_{61}^{u\dagger}s_{61}^{u} + s_{61}^{d\dagger}s_{61}^{d}s_{61}^{d\dagger}s_{61}^{d}, \quad (21)$$

that, along with Eqs. (17), (19), and (20), allows us to write

$$\langle(\Delta I)^2\rangle = \langle(\Delta I_{\mathrm{up}})^2\rangle + \langle(\Delta I_{\mathrm{down}})^2\rangle, \quad (22)$$

i.e., shot-noise current spectral densities add for mesoscopic conductors connected in parallel by means of junctions satisfying Eqs. (6).

The above demonstration has important consequences on the functionality of devices based on the Aharonov-Bohm effect, even if only one of the junction obeys to the conditions (6). Without losing generality, let us suppose that $s_{32}^{l}=0$ and $s_{23}^{l}=0$, which implies that Eq. (7) holds true.

Since paths from lead 2 to 3 passing through the left junction are not allowed, the matrix s_{61} can be written as the sum of two terms $s_{61}^{(2)}$ and $s_{61}^{(3)}$, which take into account only the paths passing through Σ^l from 1 to 2, and from 1 to 3, respectively, and can be expressed in the form

$$s_{61}^{(2)} = s_{62}s_{21}^{l}, \quad s_{61}^{(3)} = s_{63}s_{31}^{l}, \quad (23)$$

where s_{62} considers all the paths from lead 2 to 6, and s_{63} all the paths from lead 3 to 6. From Eq. (7) we have

$$s_{61}^{(2)\dagger}s_{61}^{(3)} = 0, \quad s_{61}^{(3)\dagger}s_{61}^{(2)} = 0, \quad (24)$$

which allows us to write

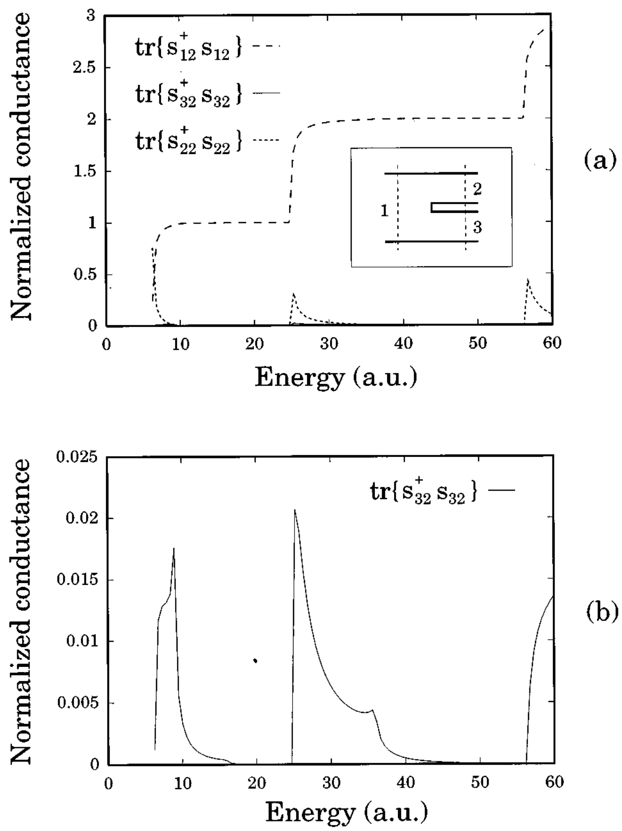

FIG. 2. The geometry of the junction is shown in the inset of Fig. (a). In (a) $\mathrm{tr}\{s_{i2}^{\dagger}s_{i2}\}$, $i=1,2,3$ are plotted as a function of the electron energy. In (b) $\mathrm{tr}\{s_{32}^{\dagger}s_{32}\}$ is plotted on a smaller scale. $\mathrm{tr}\{s_{12}^{\dagger}s_{12}\}$ has the well known staircase behavior while $\mathrm{tr}\{s_{32}^{\dagger}s_{32}\}$ is at least two orders of magnitude smaller.

$$\mathrm{tr}\{s_{61}^{\dagger}s_{61}\} = \mathrm{tr}\{(s_{61}^{(2)\dagger} + s_{61}^{(3)\dagger})(s_{61}^{(2)} + s_{61}^{(3)})\}$$

$$= \mathrm{tr}\{s_{61}^{(2)\dagger}s_{61}^{(2)}\} + \mathrm{tr}\{s_{61}^{(3)\dagger}s_{61}^{(3)}\}. \quad (25)$$

Equation (25) establishes that there is no interference between paths included in $s_{61}^{(2)}$ and in $s_{61}^{(3)}$, i.e., the associated conductances just add. In other words, devices based on the modulation of conductance due to quantum interference between different paths cannot work if just one of the junction satisfies the properties indicated in Eq. (6). This consideration has to be taken into account, for example, when devices are designed that exploit some sort of Aharonov-Bohm effect.

A junction with the properties (6) is shown in the inset of Fig. 2(a): the electron wave guide on the left (of conventional width 1) bifurcates into two channels (each of width 0.4) separated by a hard-wall barrier of width 0.2. The calculation has been performed with the mode-matching technique,[4] considering a total of 36 transverse modes. The term $\mathrm{tr}\{s_{32}^{\dagger}s_{32}\}$ is shown, on expanded scale, in Fig. 2(b).

Aharonov-Bohm devices should have not even one junction of the type shown in the inset of Fig. 2: as can be seen the term $\mathrm{tr}\{s_{32}^{\dagger}s_{32}\}$, though not exactly zero, is more than two orders of magnitude smaller than $\mathrm{tr}\{s_{12}^{\dagger}s_{12}\}$, which implies a vanishingly small quantum interference. Devices with junctions similar to that of Fig. 2 have been proposed in the literature.[17]

We have shown that conductances, thermal and shot-noise current spectral densities for parallel mesoscopic conductors add, provided a few conditions are satisfied by the junction scattering matrices. Our demonstration is trivially extended to the case of more than two conductors in parallel.

Furthermore, our considerations have important consequences on the design of Aharonov-Bohm devices, and, in particular, on the shape of the junction connecting the different paths. Finally, we have shown an example of a junction warranting additivity of conductance.

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