Università di Pisa

Weak measurement and the traversal time problem, in "Tunneling and its Implications"

Giuseppe Iannaccone
Dipartimento di Ingegneria dell'Informazione: Elettronica, Informatica, Telecomunicazioni,
Università di Pisa

G. Iannaccone, Weak measurement and the traversal time problem, in "Tunneling and its Implications", Proceedings of the Adriatico Research Conference "Tunneling and its Implications", ICTP, Trieste, July 30-August 2 1996, D. Mugnai, A. Ranfagni, L.S. Schulman Eds., World Scientific Pub., Singapore, 1997, p. 292.

Weak measurement and the traversal time problem *

G. Iannaccone†

Dipartimento di Ingegneria dell'Informazione, Università degli Studi di Pisa, Via Diotisalvi 2, I-56126 Pisa, Italy;

Institute for Microstructural Sciences, National Research Council of Canada, Ottawa, Canada K1A 0R6

Abstract

The theory of weak measurement, proposed by Aharonov and coworkers, has been applied by Steinberg to the long-discussed traversal time problem. The uncertainty and ambiguity that characterize this concept from the perspective of von Neumann measurement theory apparently vanish, and joint probabilities and conditional averages become meaningful concepts. We express the Larmor clock and some other well-known methods in the weak measurement formalism. We also propose a method to determine higher moments of the traversal time distribution in terms of the outcome of a gedanken experiment, by introducing an appropriate operator. Since the weak measurement approach can sometimes lead to unphysical results, for example average negative reflection times and higher moments, the interpretation of the results obtained remains an open problem.

I. INTRODUCTION

In the last few years a new approach to measurement in quantum mechanics has been developed by Aharonov and coworkers [1,2]. Their "weak measurement" approach differs from the standard one (formalized by von Neumann [3]) in that the interaction between the measuring apparatus and the measured system is too weak to trigger a collapse of the wave function. Although an individual weak measurement of an observable has no meaning, one can obtain the expectation value to any desired accuracy by averaging a sufficiently large number of such individual results.

Avoiding wave function collapse allows the simultaneous measurement of non-commuting observables (no violation of the uncertainty principle occurs because the individual measurements of each observable are very imprecise). It also allows a sound definition of conditional probabilities and their distribution: since the system evolves after the measurement as if unperturbed, it is possible to define averages of a quantity conditioned to a given final state of the system. Moreover – and this point is important if we are interested in the duration

*Talk given at the Adriatico Research Conference on "Tunnelling and its implications", 30 July–2 August 1996, ICTP, Trieste

of some process – a typical weak measurement is extended in time, i.e., the interaction between the meter and the system is not impulsive, but has a finite duration. As Steinberg has shown, [4,5] all these features make weak measurement theory a promising framework for the study of traversal times in quantum systems, a problem that does not fit well within standard measurement theory.

In this paper, we show that the ambiguities which are present in the formalism when the traversal time problem is studied with the tools of standard measurement theory, [6] vanish in the framework of the weak measurement approach. However, the *interpretation* of the weak measurement results remains open. The outline of the paper is as follows: In Section 2 we present briefly the weak measurement theory (WMT), in a "minimalistic" way, i.e., concentrating on only those aspects of WMT that are directly relevant to the traversal time problem. We apply the technique to this problem in Sec. 3 and in Sec. 4 show that several well known methods for defining and calculating average traversal times are particular realizations of the weak measurement approach. In Sec. 5 we go further and introduce an operator for the time spent in a region of space in an attempt to obtain higher moments of the traversal and dwell time distributions. A short discussion of open problems ends the paper.

II. WEAK MEASUREMENT: A "MINIMALIST" FORMULATION

In this section we describe the generic *gedanken* experiment and compare the standard measurement theory of von Neumann with the weak measurement theory of Aharonov and coworkers. For the scope of this paper we do not need to push the theory as far as Aharonov et al. [2] and will limit the discussion to weak measurements on an ensemble of systems, staying clear of the more controversial issues of weak measurements on a single system and the reality of the wave function (i.e., the possibility of measuring the wave function of a single system). We use a minimalist approach to weak measurement theory treating it as a potentially useful extension of standard measurement theory, based on a "weak" system-apparatus interaction Hamiltonian.

The experimental setup consists of a system Σ and a measuring device M evolving – when isolated – under the Hamiltonians $\hat{H}_\Sigma$ and $\hat{H}_M$, respectively. Let q be the canonical variable of the meter that we use as a pointer, and let π be its conjugate momentum. The corresponding operators are $\hat{q}$ and $\hat{\pi}$ with $[\hat{\pi}, \hat{q}] = -i\hbar$.

To measure an observable $\hat{A}$ of the system Σ, let the system and apparatus interact through the Hamiltonian

$$\hat{H}_{int} = g(t)\hat{\pi}\hat{A}(t), \tag{1}$$

where $g(t) = Gh(t)$, G is a constant and $\int_{-\infty}^{+\infty} h(t)dt = 1$. Let $h(t)$ be non-zero only for $t \in (t_i, t_f)$.

The system Σ and the meter M evolve independently with Hamiltonian $\hat{H}_0 = \hat{H}_\Sigma + \hat{H}_M$ until time t_i, then undergo the interaction governed by $\hat{H}_{int}$, and, after time t_f, continue their evolution under $\hat{H}_0$. What is measured is the position of the meter at time t_f.

Let us denote by $|\psi_0(t)\rangle$, $|\phi_0(t)\rangle$, and $|\Phi_0(t)\rangle \equiv |\psi_0(t)\rangle \otimes |\phi_0(t)\rangle$ the states representing the system Σ, the meter M, and their combination Σ plus M, respectively, evolving without mutual interaction, and by $|\Phi(t)\rangle$ the state of the combined system after the switching on of the interaction $\hat{H}_{int}$ at time t_i. Since the system Σ and the meter M do not interact before time t_i, $|\Phi(t)\rangle = |\Phi_0(t)\rangle$ for $t < t_i$.

For simplicity, we will consider $\hat{H}_M = 0$, that is the state of the meter is static until the interaction is turned on, so that we can use $|\phi_i\rangle = |\phi_0(t_i)\rangle$ for the state of the meter before time t_i. Moreover, after the interaction is switched off, at t_f, the state of the meter in each component of the linearly superposed entangled state no longer changes with time.

In the Schrödinger picture, [7]

$$|\Phi(t_f)\rangle = \hat{U}(t_f, t_i)|\Phi(t_i)\rangle, \tag{2}$$

where $\hat{U}(t_f, t_i)$ is the evolution operator

$$\hat{U}(t_f, t_i) = \left(\exp\left\{-\frac{i}{\hbar}\int_{t_i}^{t_f}[\hat{H}_0(t) + \hat{H}_{int}(t)]dt\right\}\right)_+, \tag{3}$$

and the $+$-subscript denotes time ordering of the integrals in the terms of the Taylor series expansion of the exponential function. In the following, we will indicate a state in the Heisenberg representation by omitting its dependence on time: for instance, $|\Phi\rangle$ is the state $|\Phi(t)\rangle$ in the Heisenberg representation, and is obtained as $|\Phi\rangle = \hat{U}(t_f, t)\Phi(t)\rangle$.

A. Standard Measurement

In the von Neumann procedure, [3] t_f tends to t_i, i.e, $h(t) \approx \delta(t - t_f)$, and what is measured is the value of the observable A at the instant of time t_f.

In the time interval (t_i, t_f), $\hat{H}_{int}$ is the dominant term in the Hamiltonian and, from (2) and (3), we have

$$|\Phi(t_f)\rangle \approx e^{-\frac{i}{\hbar}G\hat{\pi}\hat{A}(t_f)}|\Phi(t_i)\rangle. \tag{4}$$

The probability density of pointer position q after the interaction is

$$f(q) \equiv \langle\Phi|q\rangle\langle q|\Phi\rangle = \sum_n \langle\Phi|a_n, q\rangle\langle a_n, q|\Phi\rangle, \tag{5}$$

where $\{|a_n(t)\rangle\}$ is a complete set of eigenstates of $\hat{A}(t)$. Straightforward calculation [3] yields

$$f(q) = \sum_n |c_n(t_f)|^2 |\phi_i(q - Ga_n)|^2, \tag{6}$$

where $c_n(t) \equiv \langle a_n(t)|\psi_0(t)\rangle$ and $\phi_i(q - Ga_n) = \langle q - Ga_n|\phi_i\rangle$.

It is worth noticing that if the initial pointer position q is precisely defined, that is $|\phi_i(q)|^2 \approx \delta(q)$, the probability density of the final position is a sum of quasi-delta functions in one-to-one correspondence with each of the eigenvalues of $\hat{A}$.

1. Distribution of the pointer position

The first two moments of the pointer position distribution are now easy to obtain. If we take an initial distribution of q centered at $q = 0$, the mean value of q at time t_f is

$$\langle q\rangle_f \equiv \langle\Phi|\hat{q}|\Phi\rangle = \int q f(q) dq = G\langle A(t_f)\rangle, \tag{7}$$

and the mean square value of q is

$$\langle\hat{q}^2\rangle_f \equiv \langle\Phi|q^2|\Phi\rangle = \int q^2 f(q) dq = \langle\hat{q}^2\rangle_i + G^2\langle A^2(t_f)\rangle, \tag{8}$$

so that

$$(\Delta q_f)^2 \equiv \langle q^2\rangle_f - (\langle q\rangle_f)^2 = (\Delta q_i)^2 + G^2(\Delta A_f)^2, \tag{9}$$

where Δq_f, Δq_i, and ΔA_f are the standard deviations of final and initial pointer positions, and of the observable A at time t_f, respectively. The integrals without explicit limits are from $-\infty$ to $+\infty$.

2. Verification of the unperturbed state

It is interesting to calculate the probability that the state of the system Σ under observation is not changed. In order to do so, we calculate the probability P_0 of verification of the unperturbed state $|\psi_0\rangle$ at time t_f, i.e.

$$P_0(t_f) \equiv \langle\Phi|\psi_0\rangle\langle\psi_0|\Phi\rangle = \int \langle\Phi|\psi_0, q\rangle\langle\psi_0, q|\Phi\rangle dq. \tag{10}$$

If we remember that $|\psi_0(t)\rangle = \sum_n c_n(t)|a_n(t)\rangle$ we obtain

$$P_0(t_f) = \sum_{n,m} |c_n(t_f)|^2|c_m(t_f)|^2 \int \phi_i^*(q - Ga_n)\phi_i(q - Ga_m) dq, \tag{11}$$

but if $\Delta q_i \ll G\Delta a$, where Δa is the minimum difference between the eigenvalues of $\hat{A}$ ($\Delta a = \min_{n\neq m}\{|a_n - a_m|\}$), the integral in (11) is practically zero when $n \neq m$, so that we can write

$$P_0(t_f) \approx \sum_n |c_n(t_f)|^4 \leq \max_n\{|c_n(t_f)|^2\}. \tag{12}$$

Equation (12) shows that the initial state is conserved only if it is an eigenstate of $\hat{A}$; if this is not the case, the evolution of the system is strongly affected by the measurement. As will be shown in the next section, this problem does not exist in the weak measurement approach, due to the fact that the evolution of the system is perturbed only to order $o(G)$ (by $o(G)$ we mean a term such that $\lim_{G\to 0} o(G)/G = 0$).

B. Weak measurement

Weak measurement is characterized by the fact that the Hamiltonian for the interaction $\hat{H}_{int}$ is small enough to be considered as a small perturbation of the Hamiltonian $\hat{H}_0$ of the isolated system Σ, and the initial uncertainty in the position of the pointer q is much greater than G times the maximum separation between different eigenvalues of $\hat{A}$.

Most importantly, the interaction does not have to be impulsive, but can have a finite duration of time. This additional flexibility is a great advantage for measurements made over finite intervals of time.

According to perturbation theory, [8] we can write

$$|\Phi(t_f)\rangle = |\Phi_0(t_f)\rangle - \frac{i}{\hbar}\int_{t_i}^{t_f} \hat{U}_0(t_f,t)\hat{H}_{int}(t)|\Phi(t)\rangle dt, \tag{13}$$

where

$$\hat{U}_0(t_f,t) = \left(\exp\left\{-\frac{i}{\hbar}\int_t^{t_f} \hat{H}_0(t')dt'\right\}\right)_+ \tag{14}$$

is the evolution operator of the isolated system Σ. First order approximation on (13) gives

$$|\Phi(t_f)\rangle = (1+\hat{o}(G))|\Phi_0(t_f)\rangle - \frac{i}{\hbar}G\hat{\pi}\int_{t_i}^{t_f} \hat{U}_0(t_f,t)\hat{A}(t)|\Phi_0(t)\rangle h(t)dt, \tag{15}$$

where $\hat{o}(G)$ indicates a generic operator whose averages are $o(G)$.

If we introduce the hermitian operator in the Heisenberg picture

$$\mathcal{I}_H(\hat{A}) \equiv \int_{t_i}^{t_f} \hat{U}_0(t_f,t)\hat{A}(t)\hat{U}_0^*(t_f,t)h(t)dt, \tag{16}$$

we can write (15) as

$$|\Phi\rangle = [1 - \frac{i}{\hbar}G\hat{\pi}\mathcal{I}_H(\hat{A}) + \hat{o}(G)]|\Phi_0\rangle. \tag{17}$$

Now we define A_w, the *weak value* of the operator $\hat{A}$, as

$$A_w \equiv \langle\Phi_0|\mathcal{I}_H(\hat{A})|\Phi_0\rangle = \int_{t_i}^{t_f} h(t)\langle\psi_0(t)|\hat{A}(t)|\psi_0(t)\rangle dt \tag{18}$$

The probability density of q after time t_f is $f(q) \equiv \langle\Phi|q\rangle\langle q|\Phi\rangle$ and can be written, by using (17) and (18), as

$$\begin{aligned} f(q) &= \langle\Phi_0|1 + \frac{i}{\hbar}G\hat{\pi}\mathcal{I}_H(\hat{A}) + \hat{o}(G)|q\rangle\langle q|1 - \frac{i}{\hbar}G\hat{\pi}\mathcal{I}_H(\hat{A}) + \hat{o}(G)|\Phi_0\rangle, \\ &= \langle\Phi_0|\exp(\frac{i}{\hbar}GA_w\hat{\pi}) + \hat{o}(G)|q\rangle\langle q|\exp(-\frac{i}{\hbar}GA_w\hat{\pi}) + \hat{o}(G)|\Phi_0\rangle \\ &= |\phi_i(q - GA_w)|^2 + o(G). \end{aligned} \tag{19}$$

Except for terms of $o(G)$, the final distribution of pointer positions is equal to the initial one translated by G times the weak value of $\hat{A}$. It is worth noticing that if the interaction is impulsive (i.e., $h(t) \approx \delta(t - t_f)$), we have $A_w \approx \langle A(t_f)\rangle$.

1. Distribution of pointer position

The mean pointer position and the variance are, from (17) and (19), respectively

$$\langle q\rangle_f \equiv \langle\Phi|\hat{q}|\Phi\rangle = \int q f(q)dq = GA_w + o(G) \tag{20}$$

and

$$(\Delta q_f)^2 = \langle q^2\rangle_f - (\langle q\rangle_f)^2 = (\Delta q_i)^2 + o(G). \tag{21}$$

The average pointer position gives us the weak value of A; on the other hand, the variance does not give us additional information, because the weak measurement is very imprecise, due to the fact that the initial pointer distribution is very broad and the interaction is weak. Averaging over many identical experiments gives the right mean value, but does not tell us anything about the dispersion of the observed quantity, which is completely swamped by the dispersion in pointer position.

2. Verification of the unperturbed state

A fundamental property of a WM is that the evolution of Σ is practically not perturbed. In fact, verification of the initial state, using (17), yields

$$P_0(t_f) = \langle\Phi|\psi_0\rangle\langle\psi_0|\Phi\rangle = 1 + o(G) \tag{22}$$

This means that several weak measurements of different observables on a single system can be performed. As a general property, and therefore even for non commuting observables, the order of successive measurements is not important.

3. Conditional averages

While conditional averages are not well defined within standard measurement theory [6], they can be introduced in an unambiguous way within WMT, as a consequence of eq. (22) discussed above. Suppose that we want to measure the average of $\hat{A}$ conditioned to the verification of a given final state which is assumed, without loss of generality, to be a member $|\chi_n\rangle$ of an orthonormal basis $\{|\chi_n\rangle\}$, for $n = 1 \dots N$, of the Hilbert space of Σ. Since $|\chi_n\rangle\langle\chi_n|$ and $\hat{q}$ commute, we can perform a standard measurement of both of them when the interaction is over, i.e., after time t_f. Then, we keep only the readings of q corresponding to a positive verification of $|\chi_n\rangle$, and calculate the "conditional" probability distribution of the collected readings $f(q)^{(n)}$, which is of the form

$$f(q)^{(n)} = \frac{\langle\Phi|\chi_n, q\rangle\langle\chi_n, q|\Phi\rangle}{\langle\Phi|\chi_n\rangle\langle\chi_n|\Phi\rangle} = |\phi_i(q - GA_w^{(n)})|^2 + o(G), \tag{23}$$

where

$$A_w^{(n)} \equiv \frac{\langle\chi_n|\mathcal{I}_H(\hat{A})|\psi_0\rangle}{\langle\chi_n|\psi_0\rangle} = \frac{1}{\langle\chi_n|\psi_0\rangle}\int_{t_i}^{t_f}\langle\chi_n(t)|\hat{A}(t)|\psi_0(t)\rangle h(t)dt. \tag{24}$$

$A_w^{(n)}$ is the weak value of $\hat{A}$ for a system which is postselected in the state $|\chi_n\rangle$ (and preselected in the state $|\psi_0\rangle$). To order $o(G)$, the probability amplitude distribution of the meter's pointer is equal to the initial one translated by a quantity proportional to $A_w^{(n)}$. When defining A_w in (18) we did not specify a post-selected state; actually, to not perform a post-selection is equivalent to post-selecting the state $|\psi_0\rangle$ because, as we showed in (22), verification of $|\psi_0\rangle$ is positive with probabililty very close to one. Therefore, $A_w^{(n)}$ of (24) is equal to A_w of (18) to order $o(G)$ if $|\psi_0\rangle$ happens to be $|\chi_n\rangle$. If it does not, then we can write $|\psi_0\rangle = \sum_n p_n|\chi_n\rangle$, where $p_n = \langle\chi_n|\psi_0\rangle$, and have

$$A_w = \sum_n |p_n|^2 A_w^{(n)}. \tag{25}$$

It is important to notice that, while A_w is always real, $A_w^{(n)}$ is in general complex valued.

From (8) and (18) we find that the conditional average and the standard deviation of the pointer position are, respectively:

$$\langle q\rangle_f^{(n)} \equiv \int_{-\infty}^{+\infty} f(q)^{(n)} q dq = G\mathrm{Re}\{A_w^{(n)}\} + o(G) \tag{26}$$

and

$$(\Delta q_f^{(n)})^2 = \langle q^2\rangle_f^{(n)} - (\langle q\rangle_f^{(n)})^2 = (\Delta q_i)^2 + o(G), \tag{27}$$

independent of n.

In addition, from (20) and (25) we have

$$\langle q\rangle_f = \sum_n |p_n|^2 \langle q\rangle_f^{(n)}, \tag{28}$$

that is, the well known sum law of conditional probabilities holds true for pointer position readings.

III. WEAK MEASUREMENT AND TRAVERSAL TIMES

Measurement of the time duration of some process requires that the observed system and the meter interact for a finite time, a situation for which the concept of weak measurement seems to be particularly well suited. Moreover, as we have just seen, WMT could also allow us to calculate conditional averages of a given temporal quantity for various outcomes of the unperturbed system.

A well known and widely accepted result in the field of tunneling times is the dwell time, i.e. the average time spent by a particle in the region Ω irrespective of its final state. [9] If $|\psi_0\rangle$ is the state describing the particle, the dwell time in the interval (t_i, t_f) is postulated to be [10]

$$\langle t_D\rangle = \tau_D(t_i, t_f) = \int_{t_i}^{t_f} \langle\psi_0(t)|\hat{P}_\Omega|\psi_0(t)\rangle dt, \tag{29}$$

where $\hat{P}_\Omega$ is the projection operator on the region Ω. As can be seen, (29) is the mean value of $\hat{P}_\Omega$ integrated over (t_i, t_f). It is hard to imagine this time as a result of a standard measurement, because $\hat{P}_\Omega$ is not a quantum non demolition (QND) variable [11] and, if $t_1 \neq t_2$, then $\hat{P}_\Omega(t_1)$ and $\hat{P}_\Omega(t_2)$ do not commute.

However, (29) can be obtained as a result of a weak measurement. In fact, if we take $\hat{A} = \hat{P}_\Omega$, and $h(t)$ as constant in (t_i, t_f), the interaction Hamiltonian is

$$\hat{H}_{int} = Gh(t)\hat{\pi}\hat{P}_\Omega, \tag{30}$$

and from (18) we have

$$P_{\Omega w} = \frac{1}{t_f - t_i}\int_{t_i}^{t_f} \langle\psi_0(t)|\hat{P}_\Omega|\psi_0(t)\rangle dt. \tag{31}$$

Combining (29) and (31) yields the dwell time as

$$\langle t_D\rangle = \tau_D(t_i, t_f) = (t_f - t_i)P_{\Omega w} = \lim_{G\to 0}\frac{\langle q\rangle_f(t_f - t_i)}{G}, \tag{32}$$

where we have used the fact that $\langle q\rangle_f = GP_{\Omega w} + o(G)$.

Suppose we are interested in the mean time spent in Ω for some specified final state of the particle. Decomposition of dwell times in terms of particles evolving to a final state $|\chi_n\rangle$ is problematic within standard measurement theory, as has been pointed out many times: [6] the difficulty is that projection onto a region Ω and projection onto a final state $|\chi_n\rangle$ involve non commuting operators, and there are no rules uniquely specifying how to build operators for quantities involving non commuting operators (this is also the reason for conditional probabilities being problematic).

The ambiguity vanishes within the weak measurement approach: the weak value of $\hat{P}_\Omega$ for a system postselected in the final state $|\chi_n\rangle$ is, according to (24),

$$P_{\Omega w}^{(n)} = \frac{1}{\langle\chi_n|\psi_0\rangle}\frac{1}{t_f - t_i}\int_{t_i}^{t_f}\langle\chi_n(t_f)|\hat{P}_\Omega|\psi_0(t_f)\rangle dt \tag{33}$$

Therefore, the average time spent in Ω from time t_i to t_f by a particle starting in the state $|\psi_0\rangle$ and finally found in the state $|\chi_n\rangle$ is

$$\langle t_D\rangle^{(n)} = \tau_D^{(n)} \equiv \frac{(t_f - t_i)\langle q\rangle_f^{(n)}}{G} = (t_f - t_i)\mathrm{Re}\{P_{\Omega w}^{(n)}\}. \tag{34}$$

Summation over different final states holds: given $|\psi_0\rangle = \sum_n p_n|\chi_n\rangle$ then, dropping the dependence on the time interval, we can write, from (28), (32), and (34),

$$\langle t_D\rangle = \sum_n |p_n|^2\langle t_D\rangle^{(n)}. \tag{35}$$

IV. WEAK MEASUREMENT AND WELL KNOWN METHODS FOR OBTAINING TRAVERSAL TIMES

In this section we want to demonstrate that some well known approaches to the calculation of tunneling times can be seen as particular examples of weak measurement, each corresponding to a different measuring apparatus.

In particular, we will focus our attention on methods based on the Larmor clock, [9,12,13] on Feynman path-integrals, [15,18], and on absorption probabilities. [19] All of these procedures are based on the application of a small perturbation (a magnetic field, a real potential, an imaginary potential, respectively) to the region of interest. After that, the state of the particle evolves in time, and we attempt to extract the information about the time spent in the region of interest from some aspect of the perturbed wave function (the spin, the phase, or the amplitude, respectively depending on the kind of perturbation applied). In order not to perturb the evolution of the state too much, we let the perturbation tend to zero [18]. It has been demonstrated [20,21] that all the "probes" mentioned above lead to the same result.

Let us now write two formulas that will be very useful in the remainder of this section. From Appendix A, the weak value of an operator $\hat{A}$ for a system postselected in the state $|\chi_n\rangle$, defined in (24), can be written as

$$A_w^{(n)} = \frac{\partial}{\partial G}\frac{\langle\chi_n(t_f),\pi|\hat{q}|\Phi\rangle}{\langle\chi_n,\pi|\Phi_0\rangle}\bigg|_{\pi,G=0} = \frac{\langle\chi_n,\pi|i\hbar\frac{\partial}{\partial(G\pi)}|\Phi\rangle}{\langle\chi_n,\pi|\Phi_0\rangle}\bigg|_{G\pi=0}. \tag{36}$$

where the second equality is true if $\hat{q}$ can be written as $\hat{q} = i\hbar\partial/\partial\pi$ in the π-representation and $|\Phi\rangle$ depends only upon the product $G\pi$ [as it obviously does for the interaction Hamiltonian (1)].

A. Real constant potential

Let us start with a constant real potential applied only in Ω and only for $t_i < t < t_f$: the perturbation Hamiltonian is $\hat{H}_{int} = \hat{H}_V = f(t)V\hat{P}_\Omega$, with $f(t) = 1$ for $t \in (t_i, t_f)$ and zero otherwise. [18] In order to translate this perturbation into the formalism of weak measurement, we can write V in the π-representation as $V = G\pi/(t_f - t_i)$. Now the perturbative potential acting on the system Σ is of the form (30).

In this case, the weak value of the operator $\hat{P}_\Omega$ for a system postselected in the state $|\chi_n\rangle$ is, according to (36),

$$P_{\Omega w}^{(n)} = \frac{\int\langle\chi_n|\mathbf{r}\rangle\langle\mathbf{r},\pi|i\hbar\frac{\partial}{\partial(G\pi)}|\Phi\rangle d^3\mathbf{r}}{\int\langle\chi_n|\mathbf{r}\rangle\langle\mathbf{r},\pi|\Phi_0\rangle d^3\mathbf{r}}\bigg|_{G\pi=0}. \tag{37}$$

We use the convention of omitting the limits of integration when the integrals run over the whole space. Given that V is proportional to π, we can write $\Phi(\mathbf{r},V) = \langle\mathbf{r},\pi|\Phi\rangle$ and $\chi_n(\mathbf{r}) = \langle\mathbf{r}|\chi_n\rangle$, so that (34) becomes

$$\langle t_D\rangle^{(n)} = (t_f - t_i)\mathrm{Re}\{P_{\Omega w}^{(n)}\} = \mathrm{Re}\left\{\frac{\int\chi_n^*(\mathbf{r})i\hbar\frac{\partial}{\partial V}\Phi(\mathbf{r},V)d^3\mathbf{r}}{\int\chi_n^*(\mathbf{r})\Phi_0(\mathbf{r},V)d^3\mathbf{r}}\bigg|_{V=0}\right\} \tag{38}$$

Note that (38) is exactly the expression for the average time spent by a particle in the region Ω obtained by using the Feynman path-integral technique. [15] If the final state is $|\mathbf{r}\rangle$, i.e., the state corresponding to a particle found to be at $\mathbf{r}$ at time t_f, the weak value of the average time is then

$$\langle t_D \rangle^{(\mathbf{r})} = \mathrm{Re} \left\{ \frac{i\hbar}{\Phi(\mathbf{r},V)} \frac{\partial \Phi(\mathbf{r},V)}{\partial V} \right\} \Bigg|_{V=0} . \tag{39}$$

which is exactly the same expression obtained for the stay time defined in [18].

B. Pure imaginary potential

A pure imaginary potential is often used in optics to simulate the absorption of photons by a material. What happens in this case is that the probability density of the particle is not conserved, because it decreases exponentially in Ω, with a time constant proportional to the applied imaginary potential. The information about the average time spent in Ω by the particle is therefore obtained by calculating how much of the total probability has been absorbed.

The perturbation Hamiltonian in this case is [21]

$$\hat{H}_{int} = \hat{H}_\Gamma = -f(t) \frac{i\Gamma}{2} \hat{P}_\Omega \tag{40}$$

which is of the form (30) if we put $\Gamma = 2iG\pi/(t_f - t_i)$. Analogously to (37) and (38) we have

$$\langle t_D \rangle^{(n)} = \tau_D^{(n)} = - \frac{\int \chi_n^*(\mathbf{r}) 2\hbar \frac{\partial}{\partial \Gamma} \Phi(\mathbf{r},\Gamma) d^3\mathbf{r}}{\int \chi_n^*(\mathbf{r}) \Phi_0(\mathbf{r},\Gamma) d^3\mathbf{r}} \Bigg|_{\Gamma=0} , \tag{41}$$

where we have put $\Phi(\mathbf{r},\Gamma) = \langle \mathbf{r}, \pi | \Phi \rangle$. This result, again, corresponds to the one obtained in [21].

C. Magnetic Field

The well known Larmor clock method [12,13] involves applying an infinitesimal magnetic field in the z-direction, confined to the region Ω. The spin, which is initially polarized in the x-direction, precesses in the x-y plane with the Larmor frequency $\omega_L = eB/m$ when the spin is "in" Ω. The spin polarization in the y-direction plays the role of pointer position. Let us consider as the perturbation Hamiltonian only the component which acts on the spin of the particle [21]

$$\hat{H}_{int} = \hat{H}_B = f(t) \frac{\hbar \omega_L}{2} \hat{\sigma}_z \hat{P}_\Omega , \tag{42}$$

where $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ are the Pauli spin matrix operators. In this case $\hat{\pi} = \hbar \hat{\sigma}_z / 2$ acts as the pointer momentum and we put $G = \omega_L (t_f - t_i)$, so that (42) takes the form (30).

We have $\hat{\sigma}_x |\psi_0\rangle = |\psi_0\rangle$ because the initial state of the system is an eigenstate of $\hat{\sigma}_x$. From

$$[\hat{\sigma}_y, \frac{\hbar}{2} \hat{\sigma}_z] |\psi_0\rangle = i\hbar \hat{\sigma}_x |\psi_0\rangle = i\hbar |\psi_0\rangle \tag{43}$$

it immediately follows that $\hat{q} = \hat{\sigma}_y$ and $\hat{\pi} = \hbar\hat{\sigma}_z/2$ are the appropriate conjugate pointer operators. With this choice (36) becomes

$$P^{(n)}_{\Omega w} = \frac{\partial}{\partial G} \frac{\langle \chi_n, \pi | \hat{\sigma}_y | \Phi \rangle}{\langle \chi_n, \pi | \Phi_0 \rangle} \bigg|_{\pi, G=0} \tag{44}$$

and

$$\langle t_D \rangle^{(n)} = \tau^{(n)}_D = \mathrm{Re} \left\{ \frac{\partial}{\partial \omega_L} \frac{\langle \chi_n, \pi | \hat{\sigma}_y | \Phi \rangle}{\langle \chi_n, \pi | \Phi_0 \rangle} \bigg|_{\omega_L, \pi=0} \right\}. \tag{45}$$

As is easy to see by comparison with (18) of Ref. [21], expression (45) for the time spent in Ω is equal to the result obtained by Rybachenko [12] and Baz'. [13]

V. HIGHER MOMENTS OF TIME DISTRIBUTIONS

As is clear from (21) weak measurements are not useful for obtaining higher moments of a distribution for the time spent in Ω. In fact, the spread of final positions of the pointer is equal to the initial one to $o(G)$. The only way within WMT of obtaining, say, the lth order moment of an operator $\hat{A}$, is to build a meter sensitive to $\hat{A}^l$. This should have an interaction Hamiltonian of the form $\hat{H}^{[l]}_{int} = Gh(t)\hat{\pi}_l \hat{A}^l(t)$. In principle, there is no fundamental problem with this, and several meters can act simultaneously on the same system.

The crucial point is that we need to use an operator for the time spent in Ω, and not just the projector over Ω as we did in section 3. In this section we will use the "sojourn time" operator previously introduced by Jaworski and Wardlaw. [22] It is consistent with the results of section 3 and 4, and is easy to obtain from the definition of mean dwell time (29).

A. An operator for the time spent in Ω

In the Heisenberg representation, the dwell time defined by (29) can be written as

$$\langle t_D \rangle = \tau_D(t_i, t_f) = \langle \psi_0 | \hat{t}_{\Omega H} | \psi_0 \rangle \tag{46}$$

if we just define

$$\hat{t}_{\Omega H} \equiv \int_{t_i}^{t_f} \hat{U}_0(t_f, t') \hat{P}_\Omega \hat{U}^*_0(t_f, t') dt' = (t_f - t_i) \mathcal{I}_H(\hat{P}_\Omega), \tag{47}$$

In the Schrödinger representation, the operator $\hat{t}_\Omega$ corresponding to $\hat{t}_{\Omega H}$, is

$$\hat{t}_\Omega(t) = \hat{U}^*_0(t_f, t) \hat{t}_{\Omega H} \hat{U}_0(t_f, t). \tag{48}$$

For a gedanken experiment with a meter sensitive to $\hat{t}_\Omega$, the interaction Hamiltonian is

$$\hat{H}^{[1]}_{int} = G_1 h(t) \hat{\pi}_1 \hat{t}_\Omega(t), \tag{49}$$

where $h(t) = (t_f - t_i)^{-1}$ for $t \in (t_i, t_f)$, and 0 otherwise; $\hat{\pi}_1$ and $\hat{q}_1$ are the conjugate momentum and position of the meter's pointer. From (48) if follows that $\mathcal{I}_H(\hat{t}_\Omega)$ defined by (16) is equal to $\hat{t}_{\Omega H}$. Application of (18) and (24) then leads to

$$t_{\Omega w} = \langle\psi_0|\mathcal{I}_H(\hat{t}_\Omega)|\psi_0\rangle = \langle\psi_0|\hat{t}_{\Omega H}|\psi_0\rangle, \qquad t_{\Omega w}^{(n)} = \frac{\langle\chi_n|\hat{t}_{\Omega H}|\psi_0\rangle}{\langle\chi_n|\psi_0\rangle}. \tag{50}$$

If we take $\langle t_D\rangle$ defined in (29), and $\langle t_D^{(n)}\rangle$ defined in (34), we can write

$$\langle t_D\rangle = \lim_{G_1\to 0}\frac{\langle q_1\rangle_f}{G_1} = t_{\Omega w}, \qquad \langle t_D\rangle^{(n)} = \lim_{G_1\to 0}\frac{\langle q_1\rangle_f^{(n)}}{G_1} = \mathrm{Re}\{t_{\Omega w}^{(n)}\}. \tag{51}$$

As can be seen, $\hat{t}_\Omega$ leads to the same result as $\hat{P}_\Omega$, in the measurement of average traversal times.

B. Higher moments

By the means of $\hat{t}_\Omega$, we can measure any moment of order l of the distributions of times spent in Ω. We need to use a meter whose corresponding interaction Hamiltonian is of the kind

$$\hat{H}_{int}^{[l]} = G_l h(t)\hat{\pi}_l \hat{t}_\Omega^l, \tag{52}$$

where $\hat{\pi}_l$ and $\hat{q}_l$ are the operators corresponding to the conjugate momentum and position of the meter's pointer. The average of the lth power of the time spent in Ω by a particle finally found in the state $|\chi_n\rangle$ is

$$\langle t_D^l\rangle^{(n)} \equiv \lim_{G_l\to 0}\frac{\langle q_l\rangle_f^{(n)}}{G_l} = \mathrm{Re}\{\langle t_\Omega^l\rangle_w^{(n)}\}, \tag{53}$$

with

$$\langle t_\Omega^l\rangle_w^{(n)} \equiv \frac{\langle\chi_n|(\hat{t}_{\Omega H})^l|\psi_0\rangle}{\langle\chi_n|\psi_0\rangle}. \tag{54}$$

Only those pointer position readings corresponding to a postselected state $|\chi_n\rangle$ are averaged. It is worth noticing that the sum rule of conditional averages is satisfied, i.e., if $|\psi_0\rangle = \sum_n p_n|\chi_n\rangle$, then, for any integer l,

$$\langle t_D^l\rangle = \sum_n |p_n|^2\langle t_D^l\rangle^{(n)} \tag{55}$$

It is also important to point out, while $\langle t_D^l\rangle$ is positively defined, the conditional averages $\langle t_D^l\rangle^{(n)}$ are not. The lack of this important property has to prevent us from interpreting these quantities as the moments of a distribution of actual times spent by the electron in the region Ω.

C. Comparison with some results in the literature

The second moment of t_D, according to (53) and (54), is $\langle t_D^2 \rangle = \langle \psi_0 | t_{\Omega H}^2 | \psi_0 \rangle$; if we remember that $\hat{t}_{\Omega H} = \mathcal{I}_H(\hat{P}_\Omega)(t_f - t_i)$, we obtain

$$\begin{aligned} \langle t_D^2 \rangle &= (t_f - t_i)^2 \langle \psi_0 | \mathcal{I}_H^2(\hat{P}_\Omega) | \psi_0 \rangle \\ &= (t_f - t_i)^2 \int d^3r \langle \psi_0 | \mathcal{I}_H(\hat{P}_\Omega) | \mathbf{r} \rangle \langle \mathbf{r} | \mathcal{I}_H(\hat{P}_\Omega) | \psi_0 \rangle \\ &= \int d^3r |t_\Omega^{(\mathbf{r})}|^2 |\psi_0(\mathbf{r}, t_f)|^2 \end{aligned} \tag{56}$$

where, as can be easily obtained from (33) and (34), $t_\Omega^{(\mathbf{r})}$ is the weak value of the time spent in Ω by a particle finally found in $\mathbf{r}$.

Eq. (56) is essentially equal to the result obtained for the second moment of the dwell time by a few works based on the path-integral approach. [16–18,25]

We would also point out that the second moment of the time spent in Ω for a particle which is post-selected in position $\mathbf{r}$ at time t_f, i.e.,

$$\langle t_D^2 \rangle^{(\mathbf{r})} = Re \left\{ \frac{\langle \mathbf{r} | \hat{t}_{\Omega H}^2 | \psi_0 \rangle}{\langle \mathbf{r} | \psi_0 \rangle} \right\} = \frac{\langle \psi_0 | \hat{P}_\mathbf{r} \hat{t}_{\Omega H}^2 + \hat{t}_{\Omega H}^2 \hat{P}_\mathbf{r} | \psi_0 \rangle}{\langle \psi_0 | \hat{P}_\mathbf{r} | \psi_0 \rangle}, \tag{57}$$

where $P_\mathbf{r} = |\mathbf{r}\rangle\langle\mathbf{r}|$ is in general different from the time proposed in Ref. [18] on the basis of the path integral approach, that, in this formalism, would be equal to $t_\Omega^{2\,(\mathbf{r})} = \langle \psi_0 | \hat{t}_{\Omega H} \hat{P}_\mathbf{r} \hat{t}_{\Omega H} | \psi_0 \rangle / \langle \psi_0 | \hat{P}_\mathbf{r} | \psi_0 \rangle$

D. Relation between higher moments and the measurement of the first moment

In this section we show that the higher order moments of t_D obtained in Sec. 5.2 can be obtained also from the wave function $|\Phi\rangle$ of the system plus meter perturbed by the Hamiltonian for the first moment $\hat{H}_{\rm int} = G_1 h(t) \hat{\pi}_1 \hat{t}_\Omega(t)$. In fact, if we multiply both numerator and denominator of (54) by $\langle \pi | \phi_i \rangle$, and substitute (B8) in the numerator, we obtain

$$\langle t_\Omega^l \rangle_w^{(n)} = \frac{1}{\langle \chi_n, \pi | \Phi_0 \rangle} \langle \chi_n, \pi | \left(\frac{i\hbar}{\pi_1} \frac{\partial}{\partial G_1} \right)^l |\Phi\rangle \bigg|_{G_1 = 0}. \tag{58}$$

If we put $\hat{\lambda} = \hat{\pi}_1 G_1$, so that $\hat{H}_{\rm int} = \hat{\lambda} h(t) \hat{t}_\Omega(t)$, and call $\Phi(\lambda, \mathbf{r}, t_f) = \langle \pi_1, \mathbf{r} | \Phi(t_f) \rangle$ we can write for any integer l

$$\langle t_D^l \rangle^{(\mathbf{r})} = \mathrm{Re} \left\{ \frac{1}{\Phi_0(\lambda, \mathbf{r}, t_f)} \left(i\hbar \frac{\partial}{\partial \lambda} \right)^l \Phi(\lambda, \mathbf{r}, t_f). \right\} \tag{59}$$

Let us just point out that, while the form of (59) is exactly equal to the l-th complex moment of the dwell time distribution obtained on the basis of path integrals [16,17], the meaning is substantially different, since the perturbative Hamiltonian used in path-integral approaches is of the kind $\hat{H}_{\rm pi} = \hat{\lambda} h(t) \hat{P}_\Omega (t_f - t_i)$, while the perturbative Hamiltonian used for obtaining (59) is $H_{\rm int}^{[1]}$ given by (49). It is clear, for example, that the former is local in space, while the latter is not.

VI. DISCUSSION

Steinberg [4,5] has argued that weak measurement theory is a promising tool for the study of the traversal time problem. Its major advantages over the standard measurement theory are the flexibility to treat interactions between a system and a measuring apparatus that are extended in time, and the possibilty of defining conditional averages for events corresponding to non commuting operators. Both these properties are due to the fact that a weak measurement prevents the wave function of the system from collapsing.

We have shown that within WMT not only mean dwell and traversal times but also the averages of any higher powers of the time spent by particles in a region Ω, conditioned to any final state of the system, can be mathematically defined in terms of the outcome of gedanken experiments.

Unfortunately, there are severe problems of physical interpretation. As already pointed out for the special cases of the Larmor [18,23] and Salecker-Wigner clocks [26], WMT may predict negative results for the average time spent by reflected particles on the far side of a barrier. In addition, as shown here, the conditional averages of any power of the time spent in Ω are not positively defined within WMT. These unphysical results prevent us from interpreting them in terms of actual time spent by particles in the spatial region Ω.

To remain on firm ground, we are compelled to consider them as just quantities with the dimensions of time describing the response of a degree of freedom q of an apparatus to an interaction with particles that is constant in time over a finite time interval, linear, and proportional to a particle's presence in Ω. Clearly, further investigation is required to learn whether these quantities can be fruitfully used to describe the time-dependent behaviour of Σ itself, i.e., apart from the particular interaction with the meter.

VII. ACKNOWLEDGMENTS

The author would like to thank W. R. McKinnon, and B. Pellegrini, and especially C. R. Leavens for many stimulating discussions and comments on the manuscript. This work has been supported by the Italian Ministry of University and Scientific and Technological Research, by the Italian National Research Council (CNR), and by the National Research Council (NRC) of Canada.

APPENDIX A: DERIVATION OF (36)

We can start from the Eq. (24), where $A_w^{(n)}$ is defined. If we multiply both numerator and denominator by $\langle \pi | \phi_i \rangle$ for $\pi = 0$ we have

$$A_w^{(n)} = \left. \frac{\langle \chi_n, \pi | \mathcal{I}_H(\hat{A}) | \Phi_0 \rangle}{\langle \chi_n, \pi | \Phi_0 \rangle} \right|_{\pi=0} . \tag{A1}$$

Now, we have just to remember that, $\hat{1} = [\hat{q}, \hat{\pi}]/i\hbar$ and to substitute this formula into (A1) in order to obtain

$$A_w^{(n)} = \frac{1}{i\hbar}\left[\frac{\langle\chi_n,\pi|\hat{q}\hat{\pi}\mathcal{I}_H(\hat{A})|\Phi_0\rangle}{\langle\chi_n,\pi|\Phi_0\rangle} - \pi\frac{\langle\chi_n,\pi|\hat{q}\mathcal{I}_H(\hat{A})|\Phi_0\rangle}{\langle\chi_n,\pi|\Phi_0\rangle}\right]\Bigg|_{\pi=0}; \tag{A2}$$

the second term of this expression vanishes for $\pi = 0$. If we substitute (B6) for $l = 1$ into the first term to the right of (A2), we obtain Eq. (36).

APPENDIX B: A FEW FORMULAS FROM PERTURBATION THEORY

Let $|\Phi_I(t)\rangle$ and $\hat{H}_{int}^{(I)}(t)$ be the system wave function and the interaction Hamiltonian, respectively, in the interaction representation, [8] i.e.,

$$|\Phi_I(t)\rangle \equiv \hat{U}_0(t_f,t)|\Phi(t)\rangle, \tag{B1}$$

$$\hat{H}_{int}^{(I)}(t) \equiv \hat{U}_0(t_f,t)\hat{H}_{int}(t)\hat{U}_0^*(t_f,t), \tag{B2}$$

where $\hat{U}_0(t_f,t) = \exp\{-i/\hbar\int_t^{t_f}\hat{H}_0(t')dt'\}$ is the evolution operator.

From (B1) we have that $|\Phi_I(t_f)\rangle = |\Phi(t_f)\rangle = |\Phi\rangle$ and $|\Phi_I(t_i)\rangle = |\Phi_0(t_f)\rangle = |\Phi_0\rangle$, therefore

$$|\Phi\rangle = \left(\exp\left\{\frac{1}{i\hbar}\int_{t_i}^{t_f}\hat{H}_{int}^{(I)}(t)dt\right\}\right)_+|\Phi_0\rangle. \tag{B3}$$

where the +-subscript denotes time-ordering.

If we take $\hat{H}_{int} = Gh(t)\hat{\pi}\hat{A}(t)$ as given by (1), with $h(t) = (t_f - t_i)^{-1}$ for $t \in (t_i, t_f)$ and zero otherwise, and put it in (B2) and (B3), we obtain

$$|\Phi\rangle = \left(\exp\left\{\frac{G}{i\hbar}\hat{\pi}\mathcal{I}_H(\hat{A})\right\}\right)_+|\Phi_0\rangle. \tag{B4}$$

Writing the exponential in (B3) as a sum yields

$$|\Phi\rangle = \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{G}{i\hbar}\right)^m\hat{\pi}^m\left([\mathcal{I}_H(\hat{A})]^m\right)_+|\Phi_0\rangle, \tag{B5}$$

from which we obtain

$$\frac{\partial^l}{\partial G^l}|\Phi\rangle\Bigg|_{G=0} = \frac{1}{(i\hbar)^l}\hat{\pi}^l\left([\mathcal{I}_H(\hat{A})]^m\right)_+|\Phi_0\rangle, \tag{B6}$$

If we choose $\hat{A}(t) = \hat{t}_\Omega(t)$, we have the additional advantage that $\mathcal{I}_H(\hat{t}_\Omega) = \hat{t}_{\Omega H}$ does not depend on time, so that time-ordering does not matter, and we can write

$$\frac{\partial^l}{\partial G^l}|\Phi\rangle\Bigg|_{G=0} = \frac{1}{(i\hbar)^l}\hat{\pi}^l\hat{t}_{\Omega H}^l|\Phi_0\rangle, \tag{B7}$$

from which we have, after projection onto the state $|\chi_n,\pi\rangle$,

$$\langle\chi_n,\pi|\hat{t}_{\Omega H}^l|\Phi_0\rangle = \langle\chi_n,\pi|\left(\frac{i\hbar}{\pi}\frac{\partial}{\partial G}\right)^l|\Phi\rangle\Bigg|_{G=0}. \tag{B8}$$

REFERENCES

† Fax number: ++39-50-568522. Electronic address: ianna@pimac2.iet.unipi.it

[1] Y. Aharonov and L. Vaidman, Phys. Rev. A, **41** 11 (1990).
[2] Y. Aharonov, J. Anandan, and L. Vaidman, Phys. Rev. A, **47** 4616 (1993).
[3] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).
[4] A. M. Steinberg, Phys. Rev. Lett. **74**, 2405 (1995).
[5] A. M. Steinberg, Phys. Rev. A **52**, 32 (1995).
[6] S. Brouard, R. Sala, J. G. Muga, Europhys. Lett. **22**, 159 (1993).
[7] J. J. Sakurai, *Modern Quantum Mechanics* (Addison Wesley, Redwood, 1955).
[8] G. Baym, *Lectures on Quantum Mechanics* (Addison-Wesley, Redwood city, 1969), p. 247.
[9] M. Büttiker, Phys. Rev. B **27**, 6178 (1983).
[10] W. Jaworski and D. Wardlaw, Phys. Rev. A **37**, 2943 (1988).
[11] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht, 1993).
[12] V. F. Rybachenko, Yad. Fiz. **5**, 895 (1967) [Sov. J. Nucl. Phys. **5**, 635 (1967)].
[13] A. I. Baz', Yad. Fiz. **4**, 252 (1966), and **5**, 229 (1967) [Sov. J. Nucl. Phys. **4**, 182 (1966), and **5**, 161 (1967)].
[14] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965)
[15] D. Sokolovski and L.M. Baskin, Phys. Rev. A **36**, 4604 (1987).
[16] C. R. Leavens and G. C. Aers, *Scanning Tunnneling Microscopy and Related Methods*, R. J. Behm et al., eds., (Kluwer Academic Publishers, Dordrecht, 1990), p. 59.
[17] D. Sokolovski and J. N. L. Connor, Phys. Rev. A **44**, 1500 (1991)
[18] G. Iannaccone and B. Pellegrini, Phys. Rev. B **49**, 16548 (1994).
[19] R. Golub, S. Felber, R. Gähler, and E. Gutsmiedl, Phys. Lett. A **148**, 27 (1990).
[20] J. G. Muga, S. Brouard, and R. Sala, J. Phys.: Condens. Matter **4**, L579 (1992).
[21] G. Iannaccone and B. Pellegrini, Phys. Rev. B **50**, 14659 (1994).
[22] W. Jaworski and D. Wardlaw, Phys. Rev. A **45**, 292 (1992).
[23] C.R. Leavens and G.C. Aers, Phys. Rev. B **39**, 1202 (1989)
[24] C. R. Leavens, Solid State Commun. **68**, 13 (1988), **68**, ii (1988).
[25] L.S. Schulman and R.W. Ziolkowski, in *Path Integrals from meV to MeV*, edited by Sa-yakanit (World Scientific, Singapore, 1989), p. 253.
[26] C.R. Leavens and W. R. McKinnon, Phys. Lett. A **194**, 12 (1994).